THE
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EDITED BY
T. A. A. BROADBENT, M.A.
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B.B.C. TALKS FOR FIFTH FORMS: MATHEMATICS.

BY the time this number of the *Gazette* appears, most readers will know that a series of talks for Fifth Forms, under the general title, "Mathematics and Life", is being broadcast by the B.B.C. on Tuesday mornings, 11.40 a.m.–12.00 noon. The first talk was given by Mr. Siddons on April 29, and we hope to print this introductory broadcast in a future number. The full list of the talks is as follows:

April	29.	The Growth of Mathematics.
May	6, 13.	The Aeroplane and its navigation.
	20.	Mathematics in Industry.
,,	27.	Mathematics at Home.
June	3.	Mathematics as an Art.
,,	10.	Statistics: (i) How they affect Commerce.
	17.	Statistics: (ii) How they affect Government.
	24.	Mathematics and the science Laboratory.
July	1.	An examiner on examinations.

BUREAU FOR THE SOLUTION OF PROBLEMS.

APPLICANTS who have received solutions from the Bureau during the Lent Term, 1941, are requested to return them to the Secretary as soon as possible.

MATHEMATICS AND AIR NAVIGATION.

By H. T. H. Piaggio.

Probably many members of the Mathematical Association are being called upon to help with the instruction of an Air Training Corps, or of attested men who have been accepted as pilots or observers and then sent home on " deferred service ". It may be of interest to give the experiences of a teacher who came into contact with this work at an early stage and so encountered some of the inherent difficulties in their most awkward form. As Klein remarked, mathematical teaching is a function of two variables, the subject and the pupil. The difficulties with the subject are not serious, for there are few ideas in air navigation which are not already known to those who teach elementary mathematics and geography. However, these ideas are clothed in unfamiliar dress, and it is necessary to learn a few technical terms, which will be given at the end of this article.

The difficulties with the pupil are very much greater. Many of the men who are well suited physically and temperamentally for the arduous duties of an air pilot or navigator left school at fourteen years of age and have for several years worked in various occupations quite unconnected with mathematics or even arithmetic. Not only have they forgotten what they learned at school ; they have also lost the habit of study. But these obstacles, though serious, are not insuperable, as we shall see later, and the method of overcoming them raises a question of general interest in education and psychology.

The history of the instruction of members of aircraft crews may be divided into three periods. The first continued until December 1940. The second was the period of the Educational Pre-training Scheme. The third, and much the most hopeful, is that of the Air Training Corps.

The first period

For a considerable time, men accepted as air pilots were given a syllabus of elementary arithmetic, algebra, geometry, statics and dynamics, and advised to study this during the interval (usually about two months) for which they were sent home on deferred service prior to being called up. The circular stated that any of the usual elementary school books could be used, and so gave the men who had already studied the subjects at school the opportunity of continuing with the books that they were used to. Additional help came from the Regional Committees for Adult Education in H.M. Forces. Roughly speaking, there is one of these connected with each university institution. The Air Selection Board now added to its syllabus a list of the addresses of the Secretaries of these committees, with the statement that advice concerning study difficulties could be obtained from them. The committees sent out, on request, a duplicated circular containing, as a minimum, the

names of suitable textbooks. Some of them added general remarks on methods of study. At least one committee went far beyond this. By personal contact with an Initial Training Wing (the first of the four grades of training schools for air pilots), the Secretary ascertained in considerable detail the mathematical standard required, and issued a large collection of typical problems to all applicants. Moreover, he put the applicants in touch with a teacher who was willing to give free private tuition in the work required. At this time the emphasis seemed to be on trigonometry, questions on which were regularly asked by the Air Selection Board. A full course in the subject was not necessary, as the modern textbooks on Numerical Trigonometry supplied all that was required. The first student dealt with studied oblique-angled triangles and logarithms, but soon the course was restricted to right-angled triangles, sometimes with logarithms and sometimes without. The course in Applied Mathematics, which originally was of something like School Certificate standard, soon became reduced to the triangle and parallelogram of velocities. Of course, this progressive reduction in the mathematical standard required for acceptance was made in the light of experience of the Air Ministry in training. Air Force Navigational Methods have changed and the initial requirements have necessarily changed with them. An interesting case arose when two men engaged in a ladies' dress store presented themselves. Their case had the novel feature that they had been rejected by the Air Selection Board as below the minimum standard of mathematical knowledge or ability. However, they were given leave to appeal against this rejection at any time within two months. They were desperately anxious to be accepted, and willing to work several evenings a week and on Sundays. Although they had practically no knowledge of mathematics, and very little ability for the abstract side of the subject, their enthusiasm for flying was such that any information that could be seen to bear upon their goal was eagerly received. It was found that the Public Library contained several books on air navigation. Some of these were unduly restricted, as they were addressed to the civil pilot who wanted to learn to fly, in peace time, the simplest type of small aeroplane. But one book was exactly what was needed, namely, the Air Ministry's *Manual of Air Navigation*, Vol. I, usually referred to as A.P. 1234. This was the 1939 edition, which may be presumed to represent the ideas of the Air Ministry up to a date shortly before the outbreak of the war ; it is advertised for sale by at least one second-hand bookseller. The current edition cannot be obtained by the public, as it is for service use only. However, the older edition proved of great value in giving a touch of reality to the teaching. One of the drapery students made great progress and was accepted as a pilot on re-examination. He has passed through the Initial Training Wing, doing better and better in his examinations as he goes on, and is now (at the time of writing) in the Elementary Flying Training School. The other man, who

was older and had a more responsible position in business, was handicapped in his studies by having to undertake long journeys in raided areas for his firm. He had to be content with the position of Wireless Operator and Air Gunner. It may be remarked that the men themselves are usually willing to sacrifice any amount of their leisure for the training, but their employers do not always encourage them, and sometimes appear to resent any activities which might divert their attention from business, even if these are carried on after business hours.

Another remarkable case was that of an actor, aged twenty-eight, who had left school about fourteen years before. He had forgotten even elementary arithmetic, but the Air Selection Board had accepted him as a pilot, presumably on account of his dash and personality. His profession debarred him from attending evening classes, but by intensive work in the mornings and the non-matinee afternoons of a single week he acquired nearly all the arithmetic, algebra, trigonometry and triangle of velocities he needed. All this was in addition to his appearance as a compère in a revue at the local theatre! Apparently he possessed high intelligence, which was suddenly and successfully diverted to a channel unused for fourteen years.

Not all the men were as weak as these. Some had obtained a school certificate, and it was very easy to teach them all the trigonometry and air navigation they required.

The period of the Educational Pre-training Scheme

On 26th November, 1940, the Board of Education, in their Administrative Memorandum No. 261, asked the Local Education Authorities for Higher Education to provide part-time courses for men accepted as physically suitable for pilots, observers, and other members of aircraft crews, but below the basic educational standard in mathematics and English. These classes were primarily for those who otherwise would have been rejected by the Air Selection Board, but once established they have proved useful also for those who were accepted but were rather weak in mathematics. The practical effect was to raise the level of the work done in connection with Regional Committees. Air Navigation now became more studied by those fully up to basic educational standards. The Initial Training Wings, about this time, advised less emphasis on trigonometry. This scheme is still in being, and the numbers dealt with, by Technical Colleges and private tuition, are still increasing.

The Air Training Corps period

The Air Training Corps were only established on 1st February, 1941, though they incorporated the Air Cadet Corps which had been in existence for over two years. However, there has been an enormous increase in numbers. In this district the previous one or two squadrons have been replaced by eight ordinary and one university squadron. Although they have been in operation such a short time, there are already many men about to be called up

who possess much more knowledge of air matters than their predecessors of six months ago. It is reasonable to suppose that in another six months there will be an ample supply of well qualified men. The Air Ministry has issued syllabuses, detailed notes, and a comprehensive list of publications for instructors. However, for the benefit of those who are only just starting the work, it may be helpful to set down a few definitions and problems.

Technical terms and constructions

The *airspeed* of an aircraft is the speed relative to the air, or what the speed would be if there were no wind.

The *groundspeed* is the speed relative to the ground, or measured along the actual path. It depends on the airspeed and the velocity of the wind. Thus if the airspeed is 200 m.p.h. and the velocity of the wind 30 m.p.h., the groundspeed is 230 m.p.h. if the wind is directly behind, and 170 m.p.h. if the wind is dead against the aircraft. For other directions of the wind we must use the triangle or parallelogram of velocities (see the example below).

The R.A.F. always measure directions clockwise, from 0° to 360°, starting from the true north (abbreviated to *true* or to T.), the magnetic north (which in Nottingham in 1941 is about 11° west of the true north ; it varies with the locality and changes slowly each year), or the direction of the aircraft's compass (which, owing to the magnetisation of the aircraft, hardly ever points exactly to the magnetic north).

Thus East could be denoted by 90° T. or 090° T. or 101° M.,
South „ „ „ 180° T. or 191° M.,
West „ „ „ 270° T. or 281° M.

The *course* of an aircraft is the *angle* between the north and the direction in which the axis of the aircraft is pointing. This is not usually the direction in which it is actually travelling, because of the effect of the wind blowing from one side or the other.

The *track* of an aircraft is the angle between the north and the direction in which the aircraft is actually travelling.

The *drift* is the difference between the course and the track. It is measured to *starboard* (*i.e.* to the right-hand of the pilot) or to *port* (*i.e.* to his left-hand).

The following problem is of great practical importance. It is quoted from Clatworthy's *Numerical Examples in Elementary Air Navigation* (Pitman), 1940, 3s. 6d., which is a useful supplement to A.P. 1234.

"An aircraft flying at 110 m.p.h. is to make good a track of 095 true. Find the course (true) to be steered and the groundspeed if the wind is 24 m.p.h. from 023 true."

To solve this, take a point A and draw a line AN parallel to the sides of the paper, up towards the top, to represent the true north. From A draw a line AC, of indefinite length, making 95° clockwise with AN. Draw also a line AW making 23° clockwise with AN.

This is the direction *from* which the wind comes. Produce this line backwards through A to B, so that AB represents 24 m.p.h. on some convenient scale. With centre B describe a circle whose radius, to the same scale as before, represents 110 m.p.h. Let this circle cut the line AC in D. (There will be another intersection with AC produced backwards through A, but this does not concern us.) Measure the angle ADB, which is about 12°, and the length AD, which (on the scale previously used) represents 100 m.p.h. Then the drift is 12° starboard, the course 083° T., and the groundspeed 100 m.p.h.

ABD is a triangle of velocities. Mark the direction AB of the wind with a single arrow, and put another single arrow along BD, but a double arrow along AD. This reminds us that the track results from the combined action of the wind and the course. The words *wind*, *course* and *track* should be written along AB, BD and AD respectively.

If the magnetic variation is 11° W., the magnetic course corresponding to 083° T. is 094° M. To find the corresponding compass course for a particular aircraft, we consult the graph on the compass deviation card. The deviation is defined as the algebraic difference between the magnetic course and the compass course. If the deviation for a magnetic course of 094° M. is +5°, the corresponding compass course is 089° C. When the deviation is positive, the compass needle is to the East of the magnetic meridian; when negative, to the West. At some Initial Training Wings they teach mnemonics such as

"Deviation West, compass best,
Deviation East, compass least."

However, it seems much better to work from a diagram. If a man cannot remember the difference between the definitions of variation and deviation, it is doubtful whether he will be really helped by the R.A.F. mnemonic of "Cadbury's Dairy Milk Very Tasty". There is always the danger of getting it muddled, as for instance to "Terry's Milk Chocolate Very Delicious".

It may be said that the triangle of velocities is the fundamental theorem in air navigation, and many practical problems depend on it. Most of these are simple to anyone with a mathematical training, but one of a harder type is as follows:

To find the wind velocity by the reciprocal course method. Steer any course AB for t_1 seconds. If there were no wind the aircraft would come to B. Draw AD to represent the velocity W of the wind on the same scale as AB represents the airspeed A, both measured in miles per hour. Complete the parallelogram $ABCD$. Then C represents the position of the aircraft at time t_1. Now steer the "reciprocal course" CD, *i.e.* a course making an angle of 180° with the original course AB, for another t_1 seconds. Owing to the wind, the aircraft comes to E, where AD is produced to E so that AD equals DE.

Finally steer dead against the wind from E back to A. Let this take t seconds.

As $$EA = 2CB = 2Wt_1/3600,$$

and the ground speed along $EA = A - W,$

$$(A - W)\, t/3600 = 2Wt_1/3600,$$

so $$W = At/(2t_1 + t) = At/T, \text{ say,}$$

when T is the total time of flight.

The other parts of elementary air navigation may be summed up as map-reading, very elementary magnetism, and meteorology. They are explained sufficiently fully for the purpose required in the old edition of A.P. 1234. There is one point in meteorology which is of some practical importance. An altimeter, though professing to give the height of an aircraft above the ground, is really nothing more nor less than a form of aneroid barometer. If we fly from a place where the sea-level barometric pressure is 996 millibars to another where the sea-level pressure is only 967 millibars, the fall of 29 millibars will be shown on the altimeter as an increase in height of about $29 \times 30 = 870$ feet. Consequently if the altimeter is set at zero at starting and we fly to keep the reading constant at 1000 feet, our true height at the second place is really only 130 feet, which may lead to disaster if, owing to mist or darkness, the error is not discovered. The approach into a region of lower pressure can, of course, be foretold from a weather map, and it occurs (in the Northern Hemisphere) when the wind is on the port (*i.e.* the left) of the aircraft.

To conclude, it is possible that in the future air navigation at night will make more use of astro-navigation. There is a tendency to fly much higher than before, and by getting above the clouds the stars can be easily observed. To train pilots for this it would be necessary to teach them a certain amount of elementary astronomy. They could then give a new interpretation to *Per ardua ad astra*.

H. T. H. Piaggio.

University College, Nottingham.

GLEANINGS FAR AND NEAR.

1351. I have not the smallest confidence in any result which is eventually obtained by the use of imaginary symbols.—G. B. Airy, *Camb. Phil. Trans.*, Vol. 10, p. 326. [Per Dr. G. J. Lidstone.]

1352. "I must do something to rouse the spiritual side; something desperate; study something, something dry and tough. What shall it be? Theology? Algebra? What's algebra?" "It's dry and tough enough," said I; "$a^2 + 2ab + b^2$." "It's stimulating, though?" he enquired. I told him I believed so, and that it was considered fortifying to Types. "Then that's the thing for me. I'll study algebra," he concluded.

The next day . . . he got word of a young lady, one Miss Mamie McBride, who was willing and able to conduct him in these bloomless meadows.—R. L. Stevenson and Lloyd Osbourne, *The Wrecker*, chapter VII. [Per Mr. A. F. Mackenzie.]

SOME SOLVED AND UNSOLVED PROBLEMS IN THE THEORY OF NUMBERS.*

By R. Rado.

Broadly speaking, the development of mathematical thought has proceeded along two main channels. On the one hand mathematicians have been studying properties of objects which in some obvious and intuitive sense form what is called a continuum, while others were concerned with objects which one conceived as being discrete and well separated from each other. An example of the former type of mathematics is the differential calculus; and the theory of the distribution of the prime numbers 2, 3, 5, 7, 11, ... belongs to the mathematics of the discrete. The mysteries of the epsilontic—so dear to us all: "If epsilon is sufficiently small, then delta will be as small as we please"—have no direct counterpart in the theory of the natural numbers 1, 2, 3, ... The underlying idea of the continuum seems to be that of motion, while the discrete and separable appears to be more of a static nature.

In comparison the mathematics of the continuum has evolved much more powerful tools and has therefore been explored far more thoroughly and systematically than that of the discrete. There is, however, constant interplay between the two branches, and some of the most spectacular advances of our science were made when some new and unexpected link was discovered between them. It came, for instance, as a great shock to Greek thinkers when they discovered the necessity of introducing irrational numbers. This shock acted as a most powerful stimulus and challenge to their intellect and finally produced the crowning genius of Archimedes. The latest stimulating shock, through a clash between the discrete and the continuous, was caused by Georg Cantor, towards the end of last century, who proved that it is impossible to arrange the continuum of all numbers, say between 0 and 1, in a sequence $a_1, a_2, \ldots$.

While the mathematics of the continuum has more effective and general methods at its disposal, it has at the same time become much more the domain for the specialist, I mean for the professional mathematician. An intelligent non-mathematician—I earnestly hope this meeting admits the existence of such beings—might sometimes play about with whole numbers, *e.g.* derive pleasure from observing that the multiplication table of the innocent-looking number $x = 142857$ starts as follows:

$$\begin{aligned} x &= 142857, \\ 2x &= 285714, \\ 3x &= 428571, \\ 4x &= 571428, \\ 5x &= 714285, \\ 6x &= 857142. \end{aligned}$$

* A paper read to the Sheffield Branch of the Mathematical Association.

Being of a curious and inquisitive nature he might be tempted to venture a little further. On finding that

$$7x = 999999,$$

he would perhaps say to himself: " Oh, I see, that is why this multiplication table is so peculiar." At any rate, whether he understands the working of the machinery or not, he is getting an inkling of what the theory of numbers really is. How much more difficult is it to make a problem concerning, say, the differentiation of some trigonometrical function interesting to an outsider! This fact explains why the theory of numbers has always been a favourite among amateur mathematicians. It seems to be the only branch where the latter stands any chance at all of surprising the professionals with some new and interesting discovery.

Sometimes, however, his discoveries are neither new nor interesting. If I may recall a personal experience, I was once asked to go and see somebody—he happened to be an engineer—who said he had solved the problem of the primes. When I arrived he produced a huge piece of cardboard on which he had written every multiple of six and its two neighbours, up to fairly large numbers. These neighbours were written in red or blue pencil, apparently according to some pattern. His discovery was that, with the exception of two numbers, namely 2 and 3, every prime number is neighbour to a multiple of six, and all that remains to be done, he said, is to find out which neighbours are and which are not prime. It is, of course, obvious, at any rate to this gathering, that of the numbers $6n$, $6n+1$, $6n+2$, $6n+3$, $6n+4$, $6n+5$ only the second and the last have any chance of being prime numbers, the others being divisible by two or three (excluding the case $n=0$).

In dealing with natural numbers two operations are commonly employed to build up new numbers from given ones, namely addition and multiplication. Addition by itself leads to no interesting problem, multiplication by itself to conceptions such as square number, common divisor of two numbers, and in particular to that of the prime number, that is, a number which is not a product of smaller numbers. It is a useful convention not to call 1 a prime. The whole multiplicative theory of numbers is summarised in the single theorem that every positive integer can be written as a product of primes, and this representation is unique, apart from changes in the order of factors. Thus the primes are the multiplicative atoms of the world of numbers. But as soon as we consider the combined effect of addition and multiplication upon the sequence of natural numbers we are at once in deep waters. The most obvious question: Which numbers can be obtained by adding two primes? has not yet been answered. If we exclude the prime $p=2$ (it is one of the easier theorems of the subject that two is the only even prime), we see at once that we can only hope to represent as such sum p_1+p_2 even numbers from six onwards, and it has been stated, without proof, in 1742, by a certain Christian Goldbach, that in fact every

even number greater than five is the sum of two primes. The best that is known in this direction is that the probability that any even number chosen at random is a sum of two primes is equal to one. By that we mean : if a_n is the number of even numbers below $2n$ which have at least one representation as sum of two odd primes, then a_n/n approaches the limit one when n increases indefinitely.

Another attack on what is called Goldbach's Conjecture leads to the result that every number, even and odd, is sum of at most 71 primes. At any rate that was the world record up to a few years ago. I daresay it may in the meantime have been improved to, say, 69. It has also been proved that every large odd number is a sum of three primes. In this theorem ' large ' means from a certain superastronomical bound onwards.

The question which is just as obvious : Which numbers are sums of two squares, including 0^2? has found a complete answer. As in the problem relating to primes an infinity of numbers is ruled out from the very beginning. For consider the possible remainders obtained by dividing a square number by four. The formulas

$$(2n)^2 = 4n^2, \quad (2n+1)^2 = 4(n^2+n)+1$$

show that only the remainders 0 and 1 can arise. Hence the only possible remainders for a^2+b^2 are 0, 1, 2. Therefore no number of the form $4n+3$ is representable as sum of two squares. In this case no Mr. Goldbach would be so rash as to pronounce, in a tone full of conviction, the theorem that every number x except those of the form $4n+3$ is a sum of two squares. For he would at once be confronted with an imposing list of offending numbers, starting with

$$12,\ 14,\ 21,\ 22,\ 24, \ldots .$$

None of these leaves the remainder 3 when divided by 4, but at the same time they are not of the form a^2+b^2. The truth is as follows :

To decide whether a number n, say 60, is of the desired form, divide n by its largest square factor—in our case 2^2. If the result contains no prime factor of the form $4k+3$, then the representation is possible, otherwise not. We have $60/4 = 15 = 3 \cdot 5$, hence 60 is not representable, 3 being the ' offending ' prime factor. But $40 = 2^2 \cdot 2 \cdot 5$ is representable, in fact $40 = 6^2 + 2^2$.

Even if we allow sums of three squares we cannot represent all numbers. A reasoning similar to that used before shows that no number of the form $8k+7$ has a representation of the form $a^2+b^2+c^2$. One might think that this sort of thing goes on for ever, not four, not five, in fact no bounded number of squares suffices to represent all numbers. That this is not so is stated by one of the most beautiful theorems of the elementary theory of numbers (Lagrange, 1770) : Every natural number is a sum of four squares.

A similar problem presents itself when we consider cubes, fourth powers, etc., instead of squares. It can be shown that corresponding to any exponent $k = 1, 2, 3, 4, \ldots$, there exists a number s such that every positive integer is of the form

$$a^k + b^k + \ldots + f^k \quad \text{(at most } s \text{ terms)}.$$

The best possible, that is the least possible value of s, is known for every k up to 400, with the exception of $k=3, 4, 5, 6$. It is given by the greatest integer not exceeding the number $2^k+(3/2)^k-2$.

There has been a school of mathematicians who said that in proving results about natural numbers nothing should be used but natural numbers. But in spite of their war-cry : " God made the natural numbers, everything else was made by man and is therefore evil ", mathematicians quite rightly seized every weapon that was handy to assail their problems, and, particularly in recent times, applications of continuum ideas to problems concerning integers have enriched the theory of numbers very considerably.

As an example of how the continuum can be made to serve the discrete, I will write down a certain equation which can be proved by means of integration, differentiation and all the other powerful methods of analysis (it belongs to the theory of the so-called elliptic functions).

$$\begin{aligned}(x^{0^2}+x^{(-1)^2}+x^{1^2}+x^{(-2)^2}+x^{2^2}+\ldots)^4 &= 8(x+x^2+x^3+\ldots)\\ &\quad +2\,.\,8(x^2+x^4+x^6+\ldots)\\ &\quad +3\,.\,8(x^3+x^6+x^9+\ldots)+\ldots\,. \qquad |x|<1.\end{aligned}$$

Let us find the coefficient of, say, x^{333}. On the left it is exactly equal to the number of times 333 can be written as sum of four squares. On the right it is at least equal to eight. Therefore the above formula puts into evidence that 333, in fact every positive integer, is a sum of four squares. In order to prove Goldbach's Conjecture one would have to show that the development of the function

$$(x^3+x^5+x^7+x^{11}+\ldots+x^p+\ldots)^2$$

into a power series is of the form

$$ax^6+bx^8+cx^{10}\ldots,$$

where $a, b, c, \ldots$ are positive constants.

I will show how in a comparatively simple case identities between functions can actually prove quite interesting facts about integers. Let x be any number numerically less than one. The following equations are, I think, easily accepted as true and can, in fact, be rigorously established :

$$\begin{aligned}(1+x)(1+x^2)(1+x^3)\ldots &= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdots\\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)}\cdots\\ &= (1+x^1+x^{1+1}+x^{1+1+1}\ldots)(1+x^3+x^{3+3}+\ldots)(1+x^5+x^{5+5}+\ldots).\end{aligned}$$

Now compare the coefficient of x^n on either side. On the left we obviously get the number of ways in which it is possible to write n as a sum of distinct positive numbers. On the right we get the number of ways in which one can express n as a sum of positive odd numbers, this time equal numbers being admitted among the terms.

Thus we have proved that for every n the two numbers of possible representations are the same. For $n=6$, the two types of sums are:

distinct terms: $6=5+1=4+2=3+2+1$;

odd terms: $5+1=3+3=3+1+1+1=1+1+1+1+1+1$.

In both cases we have four representations.

In the examples discussed so far addition seems to play a more important part than multiplication. All such problems about the possibility or the number of representations of any given number as sum of certain specified types of numbers are grouped together under the heading of "Additive Theory of Numbers". Quite recently a new and very general approach to such questions has been found. Consider an increasing sequence of positive integers:

$$A: a_1, a_2, \ldots,$$

say 1, 7, 29, ...

We say that A has a density of at least 1/10 if, out of all numbers 1, 2, ... up to any limit x always at least $x/10$ numbers belong to A. Generally, the density is at least α if, for every x, A contains at least αx numbers not exceeding x. Consider a second sequence B: $b_1, b_2, \ldots$, of density at least equal to β. Now define $A+B$ as being the sequence of all a's, of all b's and of all a_k+b_l's. If both, A and B, are the sequence of all odd primes, then $A+B$, if Goldbach's Conjecture is true, contains all even numbers from six onwards. One might ask this question: What is the best general assertion one can make about the density of $A+B$, assuming that the density of A is at least α and that of B at least β? It has been conjectured that the density of $A+B$ is at least $\alpha+\beta$, provided, of course, this number is not greater than unity. If $\alpha+\beta$ exceeds unity it is easy to see that the density of $A+B$ is equal to one, *i.e.* that $A+B$ consists of all numbers. This seems to be a problem that does not require any elaborate technique, but it has baffled a number of the greatest mathematicians. Nobody has so far been able either to prove or, by means of some special example of sequences A and B, to disprove this so-called $(\alpha+\beta)$-conjecture. It has, however, been confirmed, by a most ingenious method, in all cases where either $\alpha=\beta$ or $\alpha=1-\beta$. All that is known in the general case, and this is very easily established, is that the density of $A+B$ is at least $\alpha+\beta-\alpha\beta$. As a matter of fact a confirmation of the $(\alpha+\beta)$-conjecture would not have any considerable effect on results concerning primes and powers since these sequences are of density zero, but the question of the density of sequences is interesting in itself, and in pure science I believe it is the intrinsic beauty and harmony that counts in the long run, not applicability.

I should like to mention another type of problem connected with sums of natural numbers. Imagine the whole system of natural numbers being divided, in an arbitrary way, into two classes A and B. A might, *e.g.*, contain all cubes and B the rest, or A: 1, 3, 6, ...; B: 2, 4, 5, ...

In fact, there is no restriction whatsoever. We ask : Can we find two numbers in A whose sum also belongs to A? Or, if not, are there two numbers in B whose sum also belongs to B? It is easily seen that one or the other of the classes always contains numbers of the required type. For consider the numbers 5, 6, 7, 8, 9 ; at least three of them must belong to the same class, say a, b, c, arranged in ascending order. Under the assumption that in no class the equation $x+y=z$ is solvable, we obtain step by step :

$$A : \ a, \ b, \ c, \quad (c-a)-(b-a)$$
$$B : \ b-a, \ c-a.$$

But then A contains b, $c-b$, c, which is a contradiction. We must note that b is different from $c-b$. For $b=c-b$ would imply that $c=2b\geqslant 10$. By an easy extension of this method we can prove more generally : however one divides the system of all natural numbers into any finite number of classes, always at least one of these classes will contain two numbers x, y, and at the same time their sum.

A rich American once asked his head gardener to plant 10 different kinds of trees along an alley in his park, which happened to be infinitely long. The trees should be equidistant. On being told that the task had been completed, the tree-lover came to inspect the alley and found to his horror that the gardener had planted the ten kinds of trees in such a way that not the slightest regularity could be discovered in the succession of the various kinds to one another. In an understandable burst of anger the man immediately sacked his head gardener and his infinitely many under gardeners, too. For the trees were of such a delicate nature that once they were taken out again they would die. On second thoughts, however, he mitigated his sentence and gave the gardener three days' time to repair the damage. This unhappy mortal racked his brains and those of his underlings until finally he found a way out. When after three days his master reappeared he found his alley bordered with trees of his favourite kind, planted in absolutely equal distances from one another and extending, to all appearances, along the entire length of the alley. The latter being infinite he had, of course, no time to check this last point, but, at any rate, he satisfied himself that there were at least $10^{10^{10}}$ trees. Asked how he had achieved this, the gardener replied : " I discovered and proved the following theorem : However you divide the sequence of all natural numbers 1, 2, 3, ... into ten or any other finite number of classes you can always find $10^{10^{10}}$ or, indeed, any other finite number of numbers which are equidistant and belong to one and the same class."

A simple application of this very beautiful theorem of modern combinatorial theory of numbers saved the gardener's job, and, I hope, has contributed to give you an idea of a sometimes sadly neglected branch of pure mathematics, the Theory of Numbers. R. R.

THE THREE HOUSES PROBLEM.

By G. T. Kneebone.

I give below a topological proof of the following theorem, based on a well-known conundrum :

A, B, C are three houses, and A', B', C' are the gas, electric and water companies, arranged as shown (Fig. 2). It is impossible to join each house to each company in such a way that no two lines cross and no line passes through a third building.

It is clear that the proof given is equally valid however the houses and companies may be arranged. Thus our problem is the same as that treated by H. W. Richmond (*Math. Gazette*, 1938, p. 180), although the two points of view are quite distinct. In order to give a rigorous mathematical treatment of a problem of this nature it is first necessary to define precisely the terms involved. Let us interpret the houses and companies as points in the plane and the lines as Jordan arcs.* In constructing our proof we may not, of course, use intuitively " obvious " properties of closed curves, etc., but we must deduce everything from the definitions we agree to adopt. This brings us into the province of Topology, and I state three standard lemmas from this subject : †

Lemma I (Jordan's Theorem).—A closed curve divides the plane into two domains of which it is the common boundary.

One of the domains is bounded, *i.e.* it is completely included in a circle of sufficiently large radius. This one is called the interior of the curve and the other is called the exterior. Suppose, now, that the plane is divided into a number of domains. Two points can then be joined by a Jordan arc which does not cross a boundary if and only if they lie in the same domain. By a closed domain I shall mean a domain together with its boundary ; thus a closed interior domain consists of all the points in and on the bounding curve. The only points " accessible " from a point within a domain are the points of the closed domain. The only points accessible from a point on a bounding curve are the points of the closed domains having the point on their boundary.

Lemma II.—Suppose j is a closed curve and b is an arc lying within j and joining two points of j. Suppose b_1 and b_2 are the two arcs into which j is divided. Then the interior of j is divided by b into two domains, one of which is the interior of $b+b_1$, the other the interior of $b+b_2$.

Lemma III.—Suppose j is a closed curve and b is an arc lying

* A Jordan curve is a curve having parametric equations $x=f(t)$, $y=g(t)$ where f, g are one-valued, continuous functions in the range $0 \leqslant t \leqslant 1$. Either no two values of t give the same point (x, y), when the curve is open and is called a Jordan arc, or there is just one pair of values, 0 and 1, of t giving the same point and the curve is closed. Curve, throughout this paper, means Jordan curve.

† For the proofs, which are long and rather delicate, see *e.g.* von Kerékjártó, *Vorlesungen über Topologie*, pp. 59–67.

outside j and joining two points of j. Suppose b_1 and b_2 are the two arcs into which j is divided. Then one of these arcs, say b_1, together with b constitutes a closed curve such that the other arc b_2 lies in its interior.

The points of a closed Jordan curve constitute a closed series (*i.e.* one without a definite beginning or end). We can define a relation of "separation of couples" among the points of such a curve in terms of the relations of magnitude between their parameters. As an example of separation we may take points of a circle. The four cardinal points may be divided into pairs in three ways. (E, W) and (N, S) separate each other but the couples (E, S) and (N, W) or (E, N) and (S, W) do not.

If two couples (P, Q) and (X, Y) on a closed Jordan curve j separate each other (Fig. 1A), then it may be shown that the two arcs into which j is divided by either couple each contain one point belonging to the other couple.

Theorem.—If (P, Q) and (X, Y) are two separating couples of points on a closed curve j, and if P, Q are joined by an arc a and X, Y are joined by an arc b in such a way that neither arc cuts the other or cuts j in a third point, then one arc lies outside j and the other lies inside.

Proof.—j divides the plane into two domains (Lemma I). Clearly each of the arcs a, b must lie entirely in one domain, and we have only to show that both cannot lie in the same one.

(a) Suppose a lies inside j. Then, by Lemma II, a divides the interior of j into two domains. The plane is then divided into three domains which may be numbered as indicated (Fig. 1A). X lies on

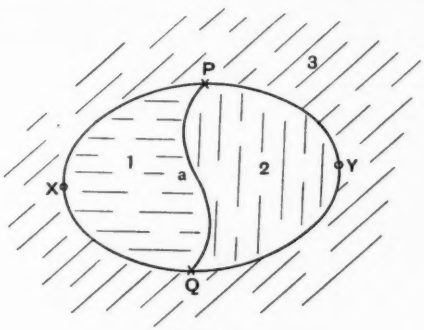


FIG. 1A.

the common boundary of 1 and 3. Hence the only points accessible from X are those of the closed domains 1 and 3. Similarly, the only points accessible from Y are those of the closed domains 2 and 3. Since the points of b are accessible from both X and Y they must all lie in the closed domain 3. Therefore in this case b is outside j.

(*b*) Suppose a lies outside j. Then, by Lemma III, a and one part of j, say a_1, together constitute a closed curve with the other part a_2 of j inside it. By Lemma II, a_2 divides the interior of this closed curve into two domains. Thus the plane is again divided into three domains (Fig 1B) and, as in (*a*), we deduce that b now lies inside j.

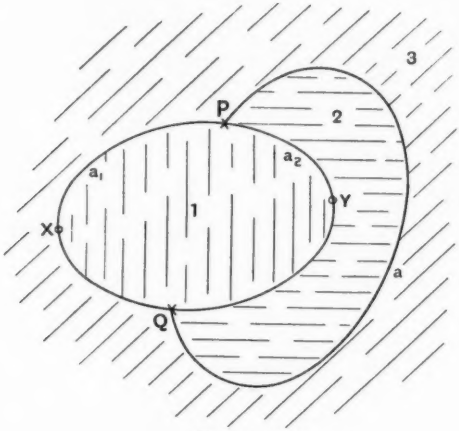


FIG. 1B.

This completes the proof of the theorem.

Let us now return to the original problem (Fig. 2), and suppose we try to make all the nine joins. If we begin by joining $CC'BA'AB'C$

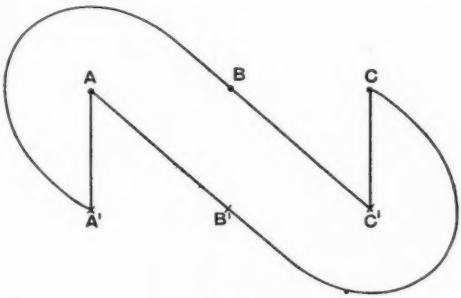


FIG. 2.

it is easy to show that we obtain a closed Jordan curve j. AC', $A'C$, and BB' remain to be joined. The figure is drawn out for greater clearness in Fig. 3.

The pairs (A', C) and (A, C') separate each other on the curve j. Hence, by the previous theorem, when they are joined one join must lie inside j and the other outside. Then, by Lemmas I-III, the plane is divided into four domains (Fig. 3). B lies on the

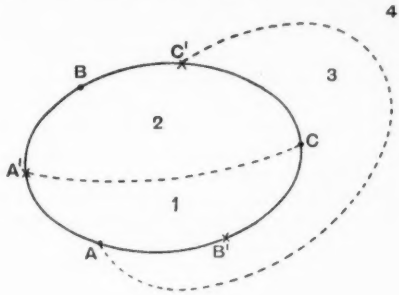


FIG. 3.

common boundary of 2 and 4, B' lies on the common boundary of 1 and 3. Therefore B' is not accessible from B, *i.e.* the remaining join cannot be made. Since it makes no difference in what order we make the various joins, the theorem stated at the beginning is proved.

The theorem is also true for six points on the surface of a sphere, as is actually the case with the houses and companies. For suppose we were able to make the nine joins on the sphere. If we chose a vertex not on any of the arcs we could derive a solution of the plane problem by stereographic projection, and this we have shown to be impossible.

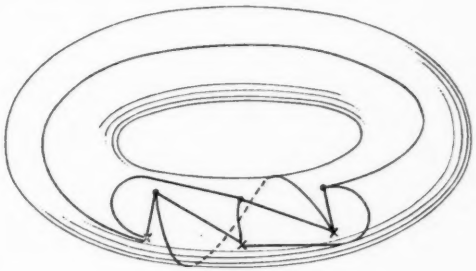


FIG. 4.

It is interesting to note, however, that the problem is soluble for points on the surface of the anchor-ring (Fig. 4). This illustrates the essential difference between the sphere and the anchor-ring, regarded as topological surfaces. G. T. K.

LONG MULTIPLICATION OF MONEY.

By H. Webb.

In a letter to the *Gazette* (XXV, February, 1941, p. 36) Mr. Siddons discussed my article in the *Gazette* for October 1940; he gave a method of long multiplication of money and suggested that I might carry out further tests using his method. I was at once attracted to Mr. Siddons' method, and carried out some tests, though I was unable to make them as extensive or as exhaustive as I desired. The present article contains a report on these tests. No very definite conclusions can therefore be drawn. This record of the work done may, however, serve as a useful pointer for those who may be sufficiently interested to carry the investigations further. I should have been pleased to have done so myself but the circumstances of my professional life have altered somewhat since carrying out the investigations recorded in recent numbers of the *Mathematical Gazette*.

The Method of Testing.

The tests were carried out with four first-year classes in a Central or Intermediate type of school. These classes are known as forms 1*a*, 1*b*, 1*c* and 1*d*. The pupils in them are approximately eleven years of age and were appointed to the classes mentioned in accordance with their ability in general school subjects. Thus the best pupils would normally be found in form 1*a* and the worst in form 1*d*.

The tests recorded below are similar to my Preliminary Tests (see *Mathematical Gazette*, October 1940). The Final Tests require a more intimate knowledge of the pupils than I possess at present. This is in order that they may be fairly divided into two equal teams. Under the circumstances the method of giving the Preliminary Tests seemed to be the best way of exercising a control.

On 16.12.40 I tested forms 1*a* and 1*d* with my own modification of the Ballard Box Method (see second example, p. 257 of the *Mathematical Gazette*, October 1940), and forms 1*b* and 1*c* with Mr. Siddons' Method. At the same time (9.30 a.m.) on 18.12.40 I reversed the tests, giving Mr. Siddons' Method * to forms 1*a* and 1*d*, and my own modification of the Ballard Box Method † to forms 1*b* and 1*c*. In both methods we commenced by placing the numerical factor above the money factor. I am so convinced of the desirability of this arrangement that I felt it might be placing a disadvantage on Mr. Siddons' Method if I did otherwise, and Mr. Siddons, himself, writes that he would not object to my arrangement.

My method of giving the tests was to collect the pupils in one group (*e.g.* 1*a* and 1*d*) together, to explain to them the method they

* Hereafter referred to as Method *S*.

† Hereafter referred to as Method *W*.

were to use, to work out one example with them according to that method and then to give them five examples to work out by themselves. The examples set are given on p. 259, *Mathematical Gazette*, October 1940. I then left supervisors with each section of the group to time the pupils as they completed the test, and I, myself, passed on to the second group (1*b* and 1*c*) and repeated the performance with the remaining method.

If a pupil had not been present for both tests his paper was discarded.

Observations on the Tests.

None of the pupils had any experience of Methods *S* or *W*. All of them were acquainted with multiplication and addition of number, and with reduction. Most of them had come across the layout of the original Ballard Box Method, but their acquaintance with it was small, and since leaving their own Junior School in July 1940 they had had little or even no experience in the use of it. The remainder of the pupils had had a similarly limited experience of the Ten-ten Method. In working the examples of the tests most pupils dealt with the reduction of farthings to pence and pence to shillings satisfactorily, but a number seemed uncertain of the method of reducing shillings to pounds. A few errors of the following order were particularly noticeable :

$$\begin{array}{r|l} 2\emptyset & 259\text{s.} \\ \hline \pounds 12 \text{ and } 1 & 9\text{s.} = 10\text{s.} \end{array}$$

This error points to insufficient experience and lack of practice.

Each incorrectly worked example was carefully examined for mistakes. There were five types of errors—errors of multiplication, addition, reduction, omission and carrying. Errors of omission occurred where pupils, when required to multiply by £56, failed to multiply by one of the numbers. Errors of carrying were recorded where pupils had failed to carry forward, or had carried forward incorrectly, the necessary results of reduction.

From the table of results it will be seen that the tests given on 18.12.40 gave better results both in speed and accuracy irrespective of the method employed. The poorer results obtained on 16.12.40 provided a number of errors which, with practice and experience, would not have occurred. This multiplicity of errors tended somewhat to obscure points which these errors would normally have made quite clear.

Until the results of the tests were finally collated I was unable to forecast how they would turn out. As I was marking the papers, however, I did gain certain impressions about the processes involved in working out the examples by the two methods. These impressions are recorded below along with the points arising out of the tests.

1. *Form*	16.12.40	18.12.40	*Improvement*	*Average*
1*a*	35·7% (*W*)	39·1% (*S*)	3·4%	37·4%
1*b*	25·6% (*S*)	49·6% (*W*)	24·0%	37·6%
1*c*	36·4% (*S*)	39·3% (*W*)	2·9%	37·9%
1*d*	33·6% (*W*)	52·0% (*S*)	18·4%	42·7%

This is a summary of certain of the results given in the complete table of results at the end of the article (p. 86).

From it will be seen that the best result was obtained with form 1*d* using Method *S*, and the greatest uplift (24%) was obtained by form 1*b* using Method *W*. Further, if the results of both tests for each class are averaged it will be found that the order of success of the classes is as follows—1*d*, 1*c*, 1*b* and 1*a*. Such apparently chaotic results may seem to indicate that the tests are not entirely reliable. In fact no finality is claimed for them, but it must be remembered that—

(i) the division of the pupils into *a*, *b*, *c* and *d* classes was on the results of a scholarship examination which included English subjects as well as Arithmetic ;

(ii) some of the best pupils may have been absent for both or one of the tests and, therefore, their papers were not included ;

(iii) " unfinished " papers were not included and there were more " unfinished " papers in 1*d* than in 1*a*. " Unfinished " papers usually belonged to the poorest pupils ;

(iv) pupils who are generally considered to be best at mathematics do not necessarily show the greatest accuracy in computation. This was true in three of the four classes tested.

2. Method *S* provided fewer errors of Multiplication than Method *W* (231 against 252). This appears to support Mr. Siddons' claim that it is an advantage to get rid of the multiplication first.

3. The advantage in multiplication of Method *S* over Method *W* was more than overcome in matters of Addition and Reduction. It was patent to me whilst marking the papers that something was going wrong with Reduction in Method *S* and it was no great surprise to me to find that it provided $33\frac{1}{3}$% more Reduction errors than Method *W*. It seemed to me that in Method *S* the more rapid passage from the conversion of one monetary quantity to that of another was the cause of the additional errors. In several instances divisors of 20 and 12 respectively were used for the reduction of pence and farthings, and in a few cases although the divisor of 20 had been written down it was obvious that a mental division by 12 had been carried out. The total number of errors of addition was small and I am at a loss to explain the disparity between the number of these errors obtained by either method. It may be that the mental energy of the pupil was largely expended in the " initial attack " by multiplication whilst the subsequent " mopping-up "

operations of Addition and Reduction were proceeded with in a more leisurely manner. This, however, can scarcely be put forward as a serious explanation unless it can be maintained that the pupil is likely to tackle each successive example with fresh or renewed vigour, for it had already been shown that in Multiplication Method *S* provided the better results.

4. The total number of errors of omission and carrying were small —45 by each method. There seemed to be no explanation for the small disparities which existed.

5. The total results indicated that Method *W* was slightly superior to Method *S* in general accuracy (200 examples correct against 193), in the time factor (3787 minutes against 4100 minutes) and in detailed mechanical accuracy (459 errors against 496). It must be added, however, that the tests cannot be regarded as sufficiently exhaustive to make a definite claim for Method *W*. I was attracted to Method *S* and so were the pupils. I was, therefore, a little surprised at the total results for, during the testing periods and during the marking, I had an impression that Method *S* would prove to be at least as efficient as Method *W*. The only point which stood out clearly during the marking was the question of Reduction (already discussed in item 3 above). My own opinion is that Method *S* is much superior to the Ten-ten Method and, when the numerical factor is placed above the money factor, it is to be preferred to the original Ballard Box Method.

It may be contended that the complete strangeness to the pupil of Method *S* may be the reason for its slightly inferior results. My own experience as an investigator into alternative methods, however, has led me to believe that a new method always gains from the stimulus of its freshness more than it loses by its strangeness.

N.B.—I have been asked why I object to labelling the columns £, s., d., f. in the second example, p. 257, *Mathematical Gazette*, October 1940. If I were to adopt such a method of labelling, the result would be

$$\begin{array}{r} 249 \\ \pounds\ \ s.\ \ d. \\ 78\ 16\ \ 8\tfrac{3}{4} \\ \hline \\ \hline \end{array}$$

I think it will be generally agreed that such an arrangement is less satisfactory than—

$$\begin{array}{r} 249 \\ \pounds 78\ 16\ \ 8\tfrac{3}{4} \\ \hline \\ \hline \end{array}$$

My objection of course does not apply to examples where the number factor is placed below the money factor.

(Table of results on p. 86.)

H. W.

Method	Method *W*				Total	Method *S*				Total
Class	1*a*	1*b*	1*c*	1*d*		1*a*	1*b*	1*c*	1*d*	
Date of Test	16.12.40	18.12.40	18.12.40	16.12.40		18.12.40	16.12.40	16.12.40	18.12.40	
No. of papers	23	25	28	25	101	23	25	28	25	101
No. of examples	115	125	140	125	505	115	125	140	125	505
No. correct	41	62	55	42	200	45	32	51	65	193
Percentage correct	35·7%	49·6%	39·3%	33·6%	39·6%	39·1%	25·6%	36·4%	52·0%	38·2%
Total time taken	928	736	898	1,225	3,787	693	1,212	1,171	1,024	4,100
Average time per paper	40·3	29·4	32·1	49	37·5	30·1	48·5	41·8	41·0	40·6
Mechanical errors of :										
Multiplication	74	56	39	83	252	51	68	76	36	231
Addition	8	11	19	7	45	12	28	14	10	64
Reduction	23	20	38	36	117	36	39	59	22	156
Omission	8	1	13	8	30	4	9	3	7	23
Carrying	5	3	2	5	15	2	11	5	4	22
Total No. of mechanical errors	118	91	111	139	459	105	155	157	79	496
Average No. of mechanical errors per paper	5·1	3·6	4·0	5·6	4·5	4·6	6·2	5·6	3·2	4·9

ARITHMETIC, ALGEBRA AND CALCULUS ON THE DRAWING-BOARD.

By W. G. Bickley.

A few sessions ago the writer was presented with the problem of providing for one hour per week a course outside the normal Intermediate syllabus in Mathematics for a group of First Year engineering students. Knowing how the engineer thinks with pencil in hand, and uses the drawing-board for a variety of design calculations, the solution adopted was that of putting as much elementary mathematics as possible on the drawing-board. This article outlines the course, along with some additional related matter, and among the reasons which seem to justify publication are

(*a*) the author—and some at least of his students!—have enjoyed the work described ;

(*b*) much of the subject-matter is not easy to find in English books ;

(*c*) a number of illuminating sidelights are thrown upon topics of the " normal " mathematics course ;

(*d*) it is believed that portions at least could be introduced with advantage into both pre- and post-certificate courses.

Geometrical Representation of Quantity.

A number or quantity can be represented by some geometrical *magnitude*, measured by an appropriate *scale*. The simplest of such representations uses segments of a straight line, measured by an ordinary graduated ruler, or by the printed lines of graph paper. Angles, or areas, may also serve, and in the former case they will be measured by means of a graduated protractor. At first uniformly spaced graduations—*uniform* scales—will be used, but it soon transpires that non-uniformly graduated scales are both necessary and effective.

Addition and Subtraction.

If numbers are represented by lengths, then addition is effected by placing appropriate line-segments end to end and in line. This may be done by using dividers to transfer the lengths from a scale, or by marking along a straight edge of paper the lengths whose sum is required, as in the method usually adopted in finding the sum of the lengths of the strips into which an indicator diagram is divided. Little more need be said about these, but there is a third method, using *two* scales, one sliding on the other. In this way we are led to the *slide-rule* as—fundamentally—an *adding* mechanism.

To perform straightforward addition (or subtraction) by any of these methods is trivial, but the slide-rule can be converted into a useful tool by the use of *functional scales*. To emphasise the fact that the slide-rule is an adding mechanism we construct slide-rules

for well-known non-linear addition formulae. Our first example is that for right-angled triangles

$$a^2+b^2=c^2. \tag{1}$$

Our scales must be such that the lengths OP, PQ, and OQ in Fig. 1, A, B, and C respectively * are proportional to a^2, b^2 and c^2. If j is the unit of length used in the graduation, we must have

$$A=ja^2, \quad B=jb^2, \quad C=jc^2.$$

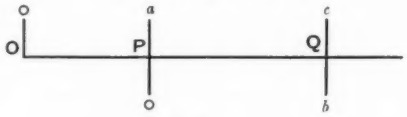


FIG. 1.

i.e. we mark off a length proportional to a^2 and label it a. We see that a and c can be measured on the same scale, while b is measured on an equal scale. In class the graduations are marked off as in Fig. 2 (a), on opposite sides of a straight line, and the paper is then cut along this line. Fig. 2 (b) shows how the two scales exhibit the well-known 3 4 5 solution of (1).

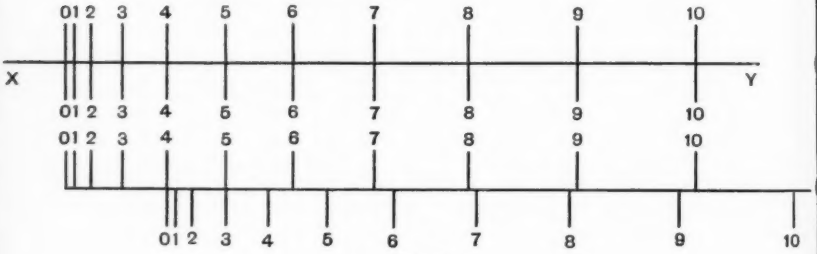


FIG. 2.

How finely the scales are graduated will depend, in class, upon the time available and the dexterity and patience of the pupils—in practice, upon the utility of the rule. It does, however, seem desirable that *one* such rule at least should be so graduated that two significant figures can be determined with certainty and a third estimated.

Similar rules are made for

$$1/u+1/v=1/f$$

and

$$v^2=u^2+64s.$$

The latter—before sliding—is shown in Fig. 3, and the method of deriving the graduation formulae is important. With

$$V=jv^2,$$

* Throughout we use capitals to denote *lengths* and smalls to denote *quantities* or graduation markings.

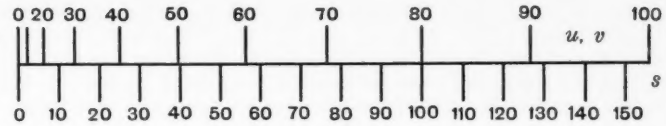


FIG. 3.

then, when $u=0$,

$$V=S,$$

or,

$$S=jv^2=j\,.\,64s,$$

so that the unit for graduating the s-scale is $64j$. The s-scale is evidently uniformly graduated, and u can be measured on the v-scale. If (as is convenient for class use) a scale 10″ long is to allow u and v to be read up to 100, $j=0{\cdot}001''$, and the unit for s is $0{\cdot}064''$, so that graduation in tens at interval $0{\cdot}64''$ is feasible.

Alternatively (especially in the case of uniform scales) we may base the graduation upon the discovery of a pair of corresponding values. Such a pair in the present case is $v=80$ when $s=100$; the subdivision may be done geometrically or by measurement.

The case of the common logarithmic slide-rule now fits into place as one instance of a general class, and not as a miracle or a special dispensation of Providence.

Geometrical Multiplication.

We next consider the possibility of drawing-board multiplication. It is common in geometry to obtain results involving products (rectangles) from equal ratios, and so frequently to employ similar triangles in deriving such results. Now the relation

$$y=ax,$$

in which we consider x as the multiplier, is expressible in terms of ratios in two ways,

$$\text{(i) } y/x=a/1 \quad \text{and} \quad \text{(ii) } y/a=x/1.$$

These lead to two families of geometrical constructions.

The disposition of the similar triangles for case (i) is shown in Fig. 4 (*a*); in virtue of the equal angles, these triangles may be superposed as in Fig. 4 (*b*). Hence we derive the first (Segner's) construction for $y=ax$.

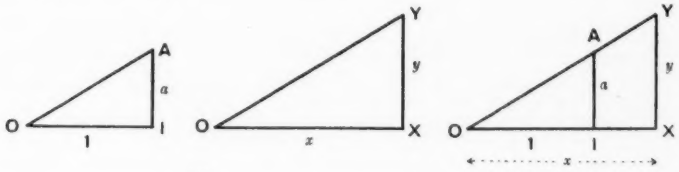


FIG. 4.

Along a line mark OI = unit and $OX = x$. Erect IA perpendicular * to OI with $IA = a$. Join OA to meet the perpendicular to OX at X in Y. Then XY represents ax on the same scale as IA represents a. (If x is very large—or very small—compared with unity, we take $OI = m$, where m is a number of the same order as x, and then—on a convenient scale—$IA = ma$.)

The diagram so obtained can, for constant a and variable x, be correlated with the "conversion" or "rate" graphs, *e.g.* the graph for converting kilograms to pounds.

Polynomials.

We are led now to consider combinations of the operations of multiplication and addition, and the simplest expression involving such a combination is

$$y = ax + b.$$

To XY we must add a length representing b. This can be done either above Y or below X. Of these the latter is the more convenient, and is equivalent to measurement from a line parallel to OX.

The construction so derived is shown in Fig. 5. OI = unit and $OX = x$. IBA is erected with $IB = b$ and $BA = a$; OP is parallel

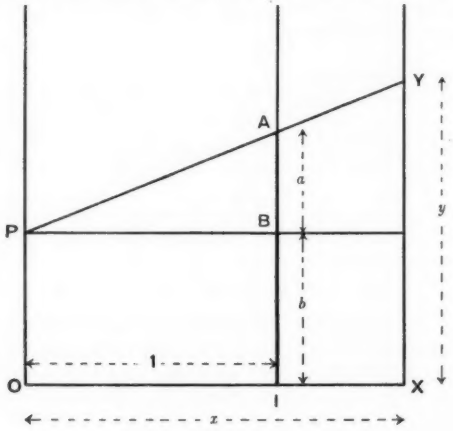


FIG. 5.

to IBA and PB to OI. PA is joined to meet the parallel to IBA through X in Y. Then XY represents $ax + b$. Among other things, this construction provides an alternative route to the fact that the graph of $y = ax + b$ is a straight line.

We can proceed further and develop a construction for

$$y = ax^2 + bx + c.$$

* The right angle is convenient, especially when squared paper is used; all that is *necessary*, however, is that XY and IA should be parallel.

We first write the quadratic in the form

$$y = (ax + b)\,x + c,$$

and see that we must treat $ax + b$ as a new multiplicand and subsequently add c. Fig. 6 shows the complete construction. $IC = c$,

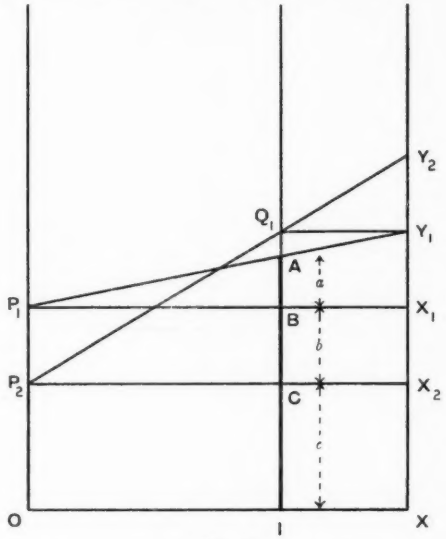


FIG. 6.

$CB = b$, and $BA = a$. $P_2P_1Y_1X_2$ corresponds to the $OPYX$ of Fig. 5; Y_1Q_1 parallel to OX transfers $ax + b\,(= X_2Y_1 = CQ_1)$ to the ordinate through I; $P_2Q_1Y_2$ effects the next multiplication by x, and c is added below X_2; thus, finally, XY_2 represents $ax^2 + bx + c$.

It should now be evident how the construction can be extended to polynomials of higher degree. Also, while all the diagrams so far given show the lay-out when x is greater than unity and all the coefficients are positive, it should be clear that the constructions are available for all values of x and of the coefficients, and the mode of dealing with negative values of one or more of these numbers should be obvious.

It should be noted that the basic ordinate—IAB in Fig. 5 or $ICBA$ in Fig. 6—is a permanent "scaffolding," for all values of x. Moreover, once this has been erected, and the points on the x-axis corresponding to the desired value or values of x have been marked off, the rest of the construction involves only parallels and concurrence; straight-edge and set-square do all the calculation and allow us to determine numerical values of an algebraic expression without doing any arithmetic.

When a series of values of x is used, the construction gives a number of points on the graph of the polynomial—we can plot the graph, using straight-edge and set-square, having measured only the coefficients along the basic ordinate. We exemplify this in Fig. 7 by obtaining points on the graph of

$$y = x^2 - 3x + 4.$$

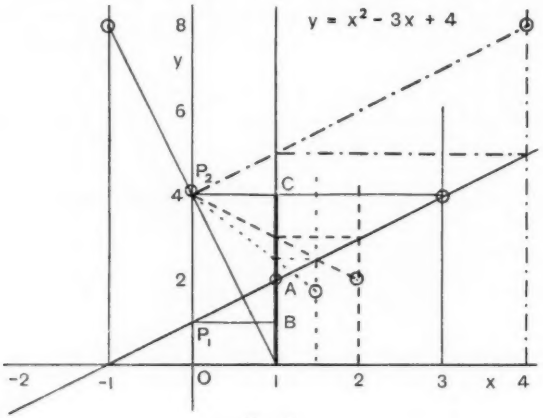


FIG. 7.

The lettering corresponds to that in Fig. 6, and the construction lines for the values 1·5, 2 and 4 of x are distinguished by different types of broken line. P_1A is, of course, common to all values of x. Note also the points for $x = 0$ and $x = 1$.

We now return to consider alternative (ii). The disposition of the quantities in the appropriate similar triangles is shown in Fig. 8,

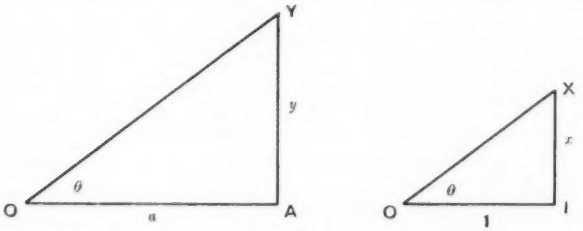


FIG. 8.

where it is seen that x now corresponds to an *angle* θ, the tangent of which is equal to x—or, to state the same facts somewhat differently, x is now represented by a *gradient*.

The easiest extension of this disposition of the quantities is to the calculation of Σax, where the a's and the x's may all be different.

Fig. 9 which, apart from the dotted lines later to be explained, should almost be self-explanatory, shows the calculation of

$$a_1x_1 + a_2x_2 + a_3x_3$$

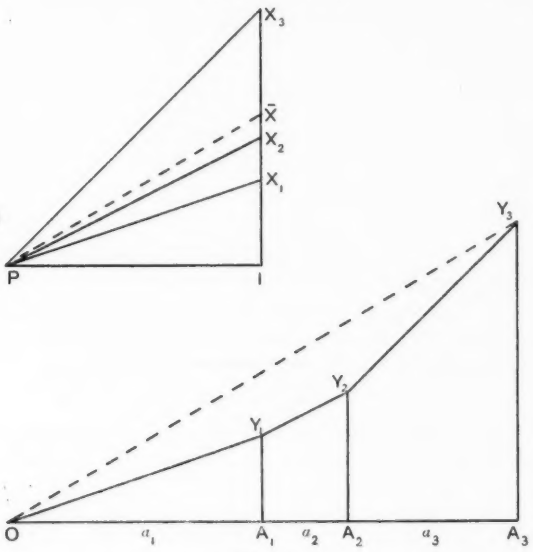


FIG. 9.

by this construction; the appropriate gradients are found by drawing the auxiliary triangle PIX_3. In the lower figure, $A_1Y_1 = a_1x_1$, $A_2Y_2 = a_1x_1 + a_2x_2$, and so on. We may further derive the (weighted) mean value of x, *i.e.* $\Sigma ax/\Sigma a$, by drawing $P\bar{X}$ in the "gradient" diagram parallel to OY_3 when $I\bar{X}$ represents $\bar{x}$.

The essentials of this construction are familiar to the engineer as the usual one for drawing bending moment diagrams due to a set of parallel concentrated loads.

We may also develop a construction (Lill's) for polynomials. Considering first $y = ax + b$, the addition of b can be effected by continuing YA in Fig. 8 for a distance b. This is done in Fig. 10 (drawn for two different values of x), where Y_1B (or $Y_1'B$) represents $ax + b$. To proceed to the quadratic $ax^2 + bx + c$, we first use the same construction to multiply $ax + b$ by x, making the angle BY_1Y_2 equal to AOY_1 (or drawing Y_1Y_2 perpendicular to OY_1), and adding c by the segment CB. The extension to polynomials of higher degree is immediate. We note again that the construction is based upon line segments representing the coefficients placed end to end, but this time not collinear.

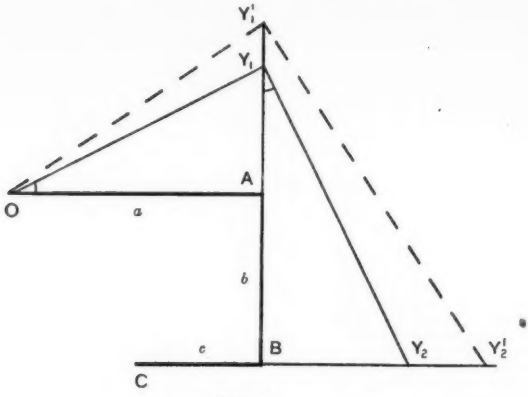


FIG. 10.

An immediate consequence of this construction is one for the solution of quadratic equations. For if

$$ax^2 + bx + c = 0,$$

then Y_2 and C must coincide. But OY_1C is a right angle, so that Y_1 must lie on a circle on OC as diameter. The construction is

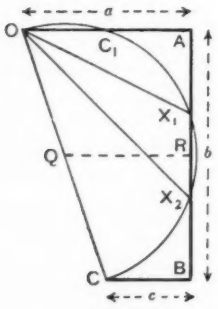


FIG. 11.

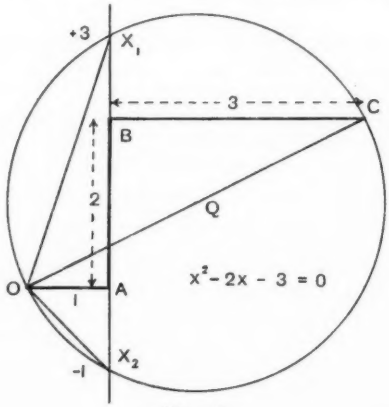


FIG. 12.

shown in Fig. 11, where the roots are given by AX_1/OA and AX_2/OA —negative in the diagram. Fig. 12 shows a case where some coefficients are negative, and gives the roots $+3$ and -1.

It is also possible to deduce from the diagram of Fig. 11 all the usual formulae for the roots of quadratic equations. Considering

AX_1 and AX_2 as lengths, and denoting as usual the roots by α and β, we have

$$AX_1 = -a\alpha, \quad AX_2 = -a\beta.$$

The point R is the mid-point both of AB and X_1X_2, so that

$$-a(\alpha+\beta) = b.$$

Again, $AX_1 \,.\, AX_2 = AO \,.\, AC_1$, and since—as is easily proved—$AC_1 = c$,

$$a^2\alpha\beta = ac, \quad \text{or} \quad \alpha\beta = c/a.$$

The condition that the roots may be real is that QR shall be less than the radius of the circle, *i.e.* that

$$\{(a+c)/2\}^2 < (b/2)^2 + \{(a-c)/2\}^2$$

or

$$4ac < b^2.$$

A cubic equation, $ax^3+bx^2+cx+d=0$, can be solved if, as in Fig. 13, points Y and Z can be found such that the angles OYZ

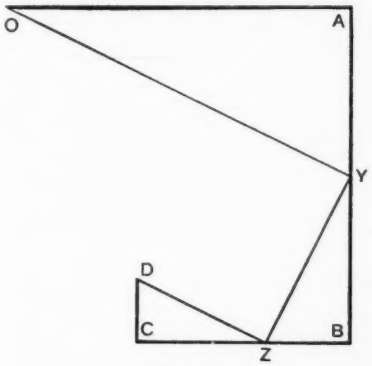


FIG. 13.

and YZD are both right angles. There is no direct "ruler and compass" construction. With luck, an approximate solution may be discovered by trial, and the process is aided by a piece of transparent squared paper which may be moved over the sheet upon which $ABCD$ is drawn.

At this point the course is varied by a "numerical interlude". The form $[\{(ax+b)\,x+c\}\,x+\ldots]$ which has proved convenient in the geometrical evaluation of a polynomial is translated into numbers, giving a technique for the systematic computation, later to be exploited in Horner's method. The question of checking a sequence of values so computed leads to finite differences; and introduction to the use of finite differences in tabulation (*e.g.* squares from the constant second difference) and interpolation (the Gregory-Newton formula) follows.

Division.

Division can be performed by inverting the process of multiplication. Fig. 4 (*b*) contains the essence of the process—a construction well known in practical geometry for subdivision of a line segment and the principle of the diagonal scale. This is satisfactory for isolated division, but a series of values of a/x, with variable x, cannot be made ordinates at abscissa x with a *fixed* unit, OI. The following construction for the more general formula

$$y = a/(x - b),$$

(Fig. 14 (*a*)) can easily be verified.

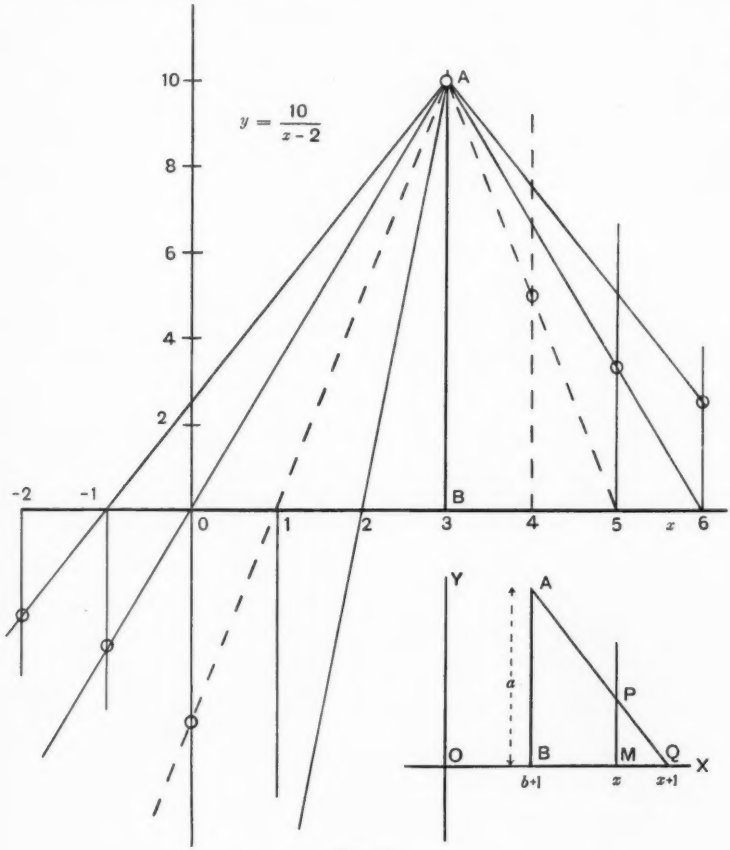


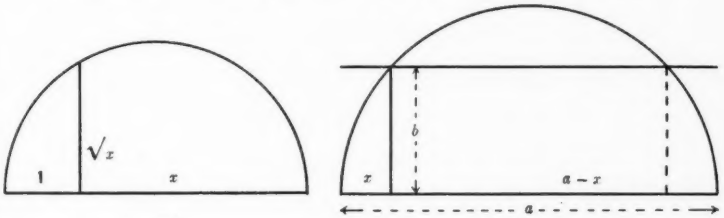
FIG. 14.

On the ordinate at B, where $OB = b + 1$, mark $BA = a$. Let Q be the point $(x + 1, 0)$. Join QA to meet the ordinate at x in P. Then $PM = a/(x - b)$.

Fig. 14 (b) shows points on the graph of $10/(x - 2)$ determined by this construction.

Roots.

The constructions so far given for powers and polynomials are not, in general, reversible. Indeed, it is known that cube and most higher roots defy "ruler and compasses". Square roots are an exception. A circle construction for square root appears in most of our books on elementary geometry—it is shown in Fig. 15 (a). It



(a) (b)

FIG. 15.

can also be derived from Lill's construction—a special case of the construction of Figs. 11 and 12 when the quadratic has no term in x.

There is also a well-known construction for the roots of a quadratic equation in the form

$$x(a - x) = b^2,$$

shown in Fig. 15 (b), and which is also given in the geometry books as a deduction from the rectangle property of chord segments. A construction, based on the tangent-secant rectangle property, exists, and is often given, for the equation of the form

$$x(a + x) = b^2.$$

Two-Variable Relations.

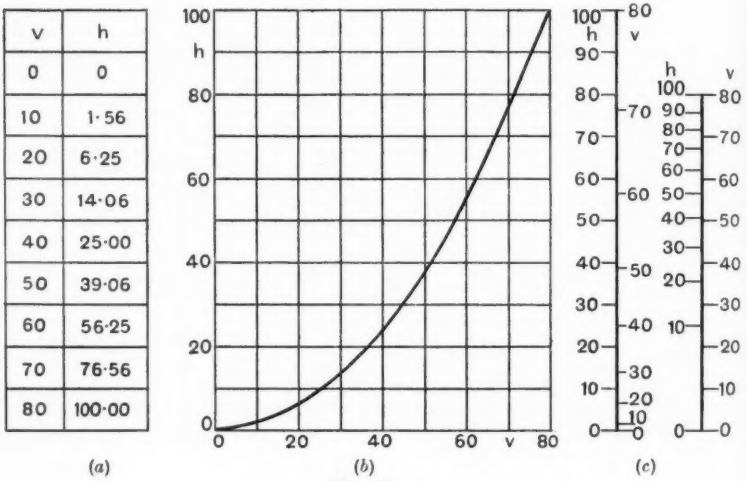
Having explored the possibilities of the geometrical performance of the elementary operations of arithmetic and algebra we next consider modes of representation of formulae—*functional relations*—such that sets of numbers which satisfy them may be determined. In the simplest case, when the formula contains but two variables, there are three ways of doing this, exhibited in Fig. 16 for the formula

$$h = v^2/64.$$

The numerical table (16a) and the ordinary graph (16b) need no comment. The use of juxtaposed scales is not so well known; it occupies much less space than the graph, but like the graph, and unlike the table, it permits interpolation by inspection. Two forms

are given in (16c), in which h and v respectively have uniform scales. Other possibilities exist—both scales logarithmic, for instance—some of which may reduce the " crowding " at one end or other of the range which these simple forms exhibit.

The principle is incorporated in the common slide-rule ; the use of the B and C scales for squares and square roots, log.-log., sine, and tangent scales, are all instances of it.



v	h
0	0
10	1·56
20	6·25
30	14·06
40	25·00
50	39·06
60	56·25
70	76·56
80	100·00

(a) (b) (c)

FIG. 16.

Three-Variable Relations.

Admitting now another variable to our formula, the numerical table becomes a table of double entry—a *set of tables*.

Corresponding to the graph, then, there is presumably a set—or *family*—of graphs. Consider the simple relation

$$y = zx.$$

For any value of z, Segner's construction gives a straight line enabling y to be read for a range of values of x. Giving z a series of values leads to a series of such lines, and so to the " graphical multiplication table " of Fig. 17. With a little practice—and to a limited accuracy—interpolation in all three directions can be carried out. Indeed, one of the exercises * on this diagram is to estimate $\sqrt{55}$.

This diagram is very simple to construct, since all the lines in it are straight, and the amount of preliminary arithmetic is negligible.

* The remark that any elaborate drawing should *always* be *used* to work some exercise or solve some problem is overdue and cannot be too strongly emphasised.

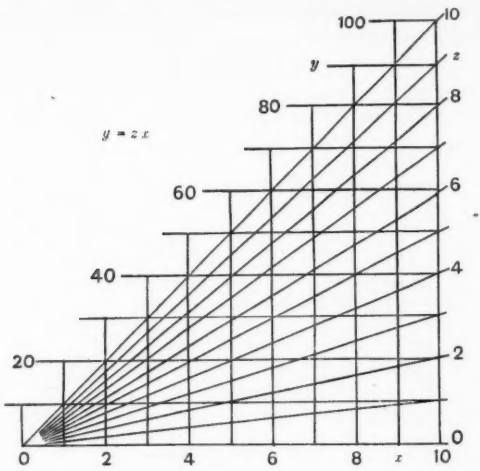


FIG. 17.

Similar treatment of a slightly less simple relation,

$$y = z\sqrt{x},$$

requires the drawing of parabolas and calculation before plotting. The result is shown in Fig. 18 (*a*). By the use of a non-uniform

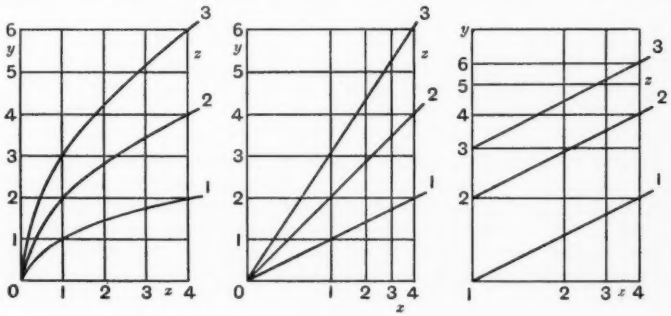


FIG. 18.

x-scale, the graphs may be straightened, as in Fig. 18 (*b*)—at the expense of the labour of graduating the non-uniform scale, and of greater difficulty and less certainty in interpolation. The use of logarithmic scales (log.-log. paper if available) also straightens the graphs, as in Fig. 18 (*c*). The point of straight graphs is, of course, that only two points on each need plotting.

An interesting and useful example is the chart, shown in Fig. 19, for the quadratic equation

$$x^2 + px + q = 0.$$

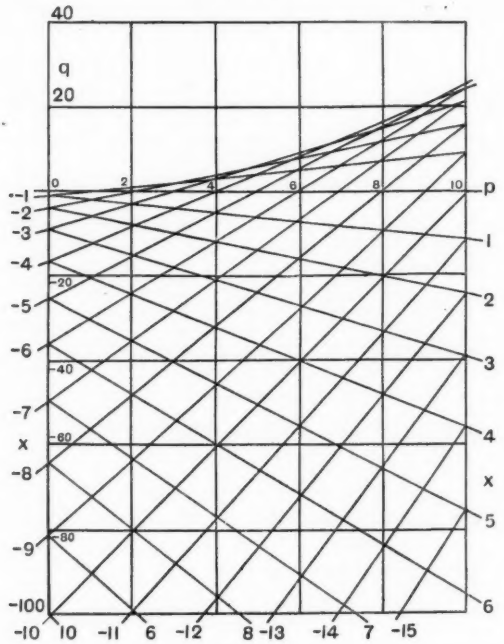


FIG. 19.

Taking p and q as rectangular co-ordinates, a straight line corresponds to any particular value of x. Through any point below the envelope, two such lines pass, corresponding to the two roots of the appropriate quadratic. Above the envelope there are no lines and so no (real) roots ; along it the roots are equal. The equation of the envelope can be shown (or, with a junior class, verified) to be

$$p^2 = 4q,$$

so that the condition for real roots is

$$p^2 > 4q.$$

(The diagram is drawn for positive p only. Changing the sign of p changes the sign of both roots.)

A similar diagram for the cubic,

$$x^3 + px + q = 0,$$

shows no blank region, but the p-q plane is divided by the semi-cubical parabola

$$4p^3 + 27q^2 = 0$$

into regions where there are three or one real roots respectively.

Instances from without the course of the principle here displayed are isothermals of a gas, equipotentials and lines of force, and contours.

There is, of course, the representation of a three-variable relation by means of a surface. Methods of descriptive geometry permit the representation of a surface in two dimensions in various ways, such as by contours—equivalent to the foregoing—or by isometric or some similar projection. We give no example of this ; such diagrams are qualitative rather than quantitative, since interpolation is difficult. At the same time, examples such as those to be found in Jahnke and Emde's *Funktionentafeln* are very attractive and highly instructive.

For many relations between three (or more) variables, nomograms are possible. The course contains an introduction to the design of the parallel scale (addition) and N or Z (multiplication) types, and others are exhibited. Since literature on nomograms is available and easily accessible, we omit details. It should be clearly understood, however, that the work on nomograms is regarded as among the most important in the course, and some of the instances of non-uniform scales previously given have been incorporated largely in order that the students should be familiar with the use of such scales before their use in connection with the collineation principle of the nomogram. Attempts to make students assimilate two new ideas at once are apt to be worse than merely unsuccessful!

Graphical Integration.

Integration is more important as regards its applications than differentiation, and from our present standpoint this is fortunate, since graphical (like numerical) integration is susceptible of a relatively high degree of accuracy, while per contra—and in consequence!—high accuracy with graphical (or numerical) differentiation is impossible.

The representation of a definite integral by the area under a curve, coupled with the graphical evaluation of definite integrals, nowadays forms part of any elementary course on calculus. If equi-distant ordinates are readily available, Simpson's rule is to be advocated ; it makes most efficient use of the data, without the labour of drawing, and is a mechanical process economising thought. But with experimental data at unequal intervals the graphical method for estimating the average height of each strip is convenient and quite as accurate as the data warrant.

For continuous integration—the construction of the graph of

$$I(x) = \int^x y(x)\,dx$$

from the given graph of $y(x)$—there is a useful and convenient, but

comparatively little known, method, exemplified in Fig. 20. Its derivation or explanation can be based upon the construction of Fig. 9, or upon the funicular polygon, or can be independently built up.

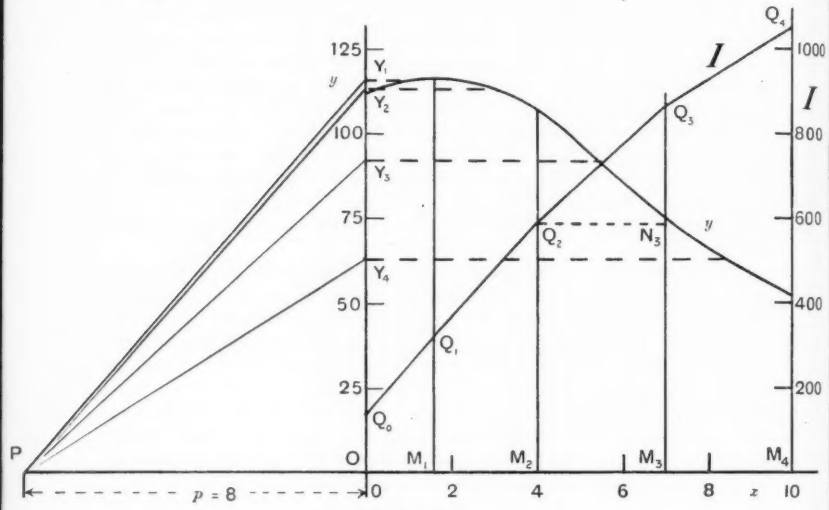


FIG. 20.

Suppose that the construction has proceeded as far as the point Q_2 on the integral curve. The average gradient of the integral curve in the next segment M_2M_3 must be proportional to the mean height $\bar{y}$ of the given curve in this interval. The gradient is obtained from this mean height by the use of some common base. To do this conveniently the estimated mean height is transferred to OY; it is OY_3 in the case of our strip. A segment OP (P is the *pole*) of length (*polar distance*) p is the common base. PY_3 now has the requisite gradient, so we draw Q_2Q_3 parallel to PY_3 to meet the ordinate at M_3 in Q_3. The process proceeds thus step by step, from a starting point Q_0 determined by the initial value I_0. $Q_0, Q_1, Q_2 \ldots$ lie on the integral curve. With narrower strips the "polygon" effect is hardly noticeable; our diagram has been kept "open" for clarity.

The vertical scale for I must, however, be determined before I_0 can be plotted (unless it is zero!), or in any case before numerical values can be read off. Now

$$\delta I = \bar{y}\,\delta x,$$

and by similar triangles

$$N_3Q_3/Q_2N_3 = OY_3/PO,$$

so that

$$\delta I = \bar{y}\,\delta x = p\,.\,N_3Q_3.$$

Thus, p being measured on the x-scale, the increment of the integral is measured by p times the corresponding interval on the y-scale. To graduate the I-scale we therefore have to multiply by p the numbers on the y-scale.

The process possesses considerable flexibility. The widths of the strips need not be equal, but may be chosen according to the rapidity of variation of y, or the difficulty of estimating mean heights. The I-scale is pre-determined so as to be easy to read and also so that the I-graph uses to the best advantage the available paper, and the necessary value of p calculated. Thus in Fig. 20 the average value of y is somewhere near 90, so that the integral amounts to about 900 ; adding in the initial value of I it appears that we must arrange for about 1000 in a distance of about 125 y-units. 8 and 10 are suitable values of p.

Indeed, the polar distance can be chosen so that it performs additional arithmetical operations, such as conversion of units. Thus if a train of mass 600 tons, starting from rest, is subject to a net accelerating force in the first 30 sec. of its motion, given in the table

t (sec.) - -	0	5	10	15	20	30
F (tons wt.) -	90	62	45	34	26	15

we can construct a graph enabling the velocity in m.p.h. to be read off on a scale *easy to read*. If v is this velocity,

$$v = \frac{60}{88}\,\frac{32}{600}\int_0^t F\,dt.$$

From the F-t graph (or even direct from the table) we can estimate the average force as about 40 tons wt., giving a final velocity of about 44 m.p.h. Thus a suitable v-scale would have 45 m.p.h. opposite 90 tons wt. The value of p to effect this (*accurately*) is determined by the equation

$$45 = 90 \times \frac{60}{88} \times \frac{32}{600} \times p$$

(which is built up by applying the succession of conversion factors to the 90) ; from this

$$p = 13{\cdot}75.$$

The geometry now does both the integration and the additional arithmetic.

Graphical Differentiation.

The above construction can be inverted, starting with the curve $Q_0Q_1 \ldots Q_n$, to produce the graph of its derivative. It is probably less inaccurate to use the gradient of a chord associated with its mid-abscissa than to attempt to draw tangents ; it is, however, *not* necessary that consecutive chords should be coterminous.

Drawing-Board Solution of Differential Equations.

Although not First Year work, our final example is in the spirit of the foregoing, and has been part of our Third Year course for some years.

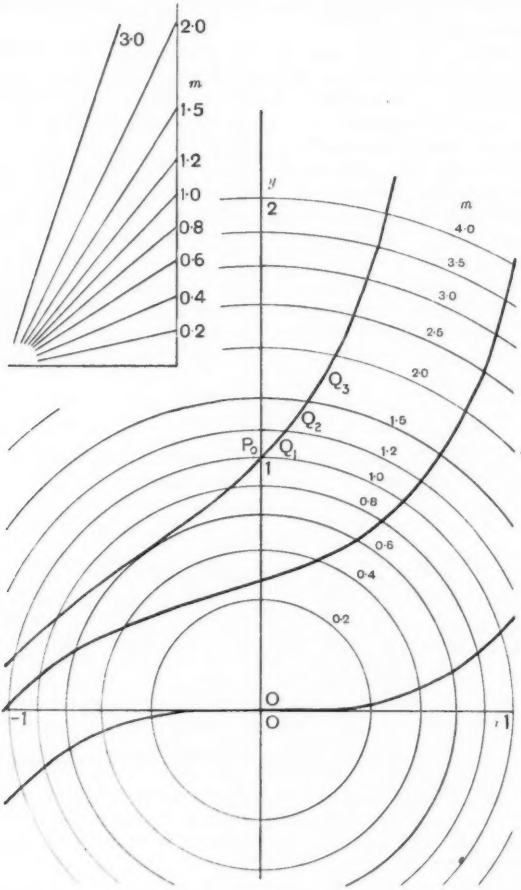


FIG. 21.

The method of isoclinals is described in many texts on differential equations as a means of obtaining by a rough sketch the general trend of the family of solutions of a first order equation. It does not seem generally to be realised that, with more careful drawing,

graphs of particular solutions can be obtained of an accuracy comparable with that of other drawing-board methods.

In Fig. 21 the process is applied to three integral curves of the equation

$$dy/dx = x^2 + y^2,$$

passing through the points (0, 1), (0, 0·5) and (0, 0) respectively. The isoclinal corresponding to a gradient m is the circle of radius $\sqrt{m}$; isoclinals are drawn for a range of values of m. The point P_0 (0, 1) lies on the isoclinal for which $m = 1$, and we draw a portion of the integral curve—or rather its tangent—through P_0 with this gradient, stopping at Q_1 half-way to the next isoclinal. Then we draw Q_1Q_2 with gradient associated with this next isoclinal, across it and half-way to the succeeding one, and so on. If " half-way " needs more precise definition, it may be taken as " to the isoclinal with the mean gradient ". The " gradient diagram " shown in the top left is not necessary when squared paper is being used.

The accuracy of individual drawings, and the consistency as regards results by a group of students, have surprised the present writer. W. G. B.

1353. The results given are the same in both editions [of *Harmonics*, by Dr. Robert Smith], but the improvements in the second edition add considerably to the learned obscurity in which the subject is involved. . . . For myself, I made out what ought to have been the theory from the *formulae*, and then was successful in mastering Smith's explanations.—Aug. De Morgan, *Camb. Phil. Trans.*, Vol. 10, p. 129. [Per Dr. G. J. Lidstone.]

1354. In a survey taken within the last week, the British Institute of Public Opinion has discovered that no less than 84%, nearly nine out of ten, of the public approve of inoculation against epidemic disease as a preventive. . . . The answers were recorded as follows :

Should be made compulsory and free	33%
Should be made free but not compulsory	51%
Is unnecessary	10%
Don't know	6%

—*News Chronicle*, 20/11/1940. [Per Mr. J. W. Ashley Smith.]

1355. A reporter suddenly finding himself in the midst of the peaceful group would have been overjoyed. He would have had to take great care not to step on some member of the Institute of France, lazily stretched out on the ground, or not to kick a Nobel Prize winner. Intellect was abundantly represented. . . . If you wanted to talk physics there were Jean Perrin, Marie Curie, André Debierne, Vida Auger. Mathematics, integral calculus? Apply to Émile Borel, draped in his bath-robe like a Roman emperor in his toga.—E. Curie, *Madame Curie* (English translation, 1939), p. 326.

1356. While it is no doubt impressive that, say, electromagnetic waves occur in Nature independently of man's interference, it is even more impressive that prime numbers, reckoned alternately, should fall into two sets, one of which gives the remainder 1, and the other 3, when divided by 4.—Review of G. H. Hardy's *Mathematician's Apology* in *Truth*, November 29, 1940. [Per Dr. H. Jeffreys.]

CORRESPONDENCE.

The Hamiltonian Revival.

To the Editor of the *Mathematical Gazette*.

Sir,—In his fascinating article "The Hamiltonian Revival", in your issue of July last, Professor E. T. Whittaker turned aside to deal an uncalled-for stroke against two great men and part of their work in the following passage:

> "Then those who were in the outer circles of Hamilton's influence—*e.g.* Willard Gibbs in America and Heaviside in England—wasted their energies in devising bastard derivatives of the quaternion calculus—dyadics and vector-analysis—which reproduced its most regrettable feature, namely the limitations imposed by its close association with the geometry of ordinary space, and which represented no advance, but rather a retrogression, from the point of view of general theory."

Doubtless many others besides myself read these words with surprise and regret, because no one can be unaware of the growing appreciation and use of vectors (and to a less extent of dyadics), both as a means of expression and as a tool in mathematical technique.

The word *bastard* may be used in either a technical sense or merely as a term of abuse, but its user may reasonably be asked to justify its application, and also the charge of *wasted* energies. Professor Whittaker indeed indicates what he regards as the "most regrettable feature" of vector analysis and dyadics, and I hope to comment on that criticism; but before doing so it would be well to know whether this is the sole justification of his condemnatory words, or if not, what are the other counts in his indictment.

E. A. Milne.

To the Editor of the *Mathematical Gazette*.

Sir,—May I take the occasion presented by Professor Milne's letter to explain more fully what was in my mind when I wrote the sentence he objects to.

First, let me recall a definition: Any set of objects of thought may be called *generalised numbers*, provided we can define what is meant by the "equality" of any two of them, and can also define the operations of "addition" and "multiplication" with respect to them, and provided also that the set of objects form a "group" with respect to these operations, *i.e.* the effect of combining two of the objects by means of one of the operations is to produce an object which also belongs to the set. The operations should satisfy certain conditions which need not be given here.

Thus, matrices of any order n may be regarded as generalised numbers; for we can define the equality, addition, and multiplication of matrices; and the result of adding or multiplying two matrices is to produce another matrix.

Another example of generalised numbers is quaternions; for we can define the equality, addition, and multiplication of quaternions; and the result of adding or multiplying two quaternions is to produce another quaternion.

Whenever we have a set of generalised numbers, we can work out a calculus based on them.

Now vectors are *not* generalised numbers; for the product of two vectors is not a vector. It is true that in vector-analysis we meet with a quantity called a vector-product of two vectors (namely, a vector at right angles to their plane and proportional to the area of the triangle formed by them), but this has not the properties of a true product, since if $\vec{\alpha\beta}$ denotes the vector-product of $\vec{\alpha}$ and $\vec{\beta}$, then the equation $\vec{\alpha\beta}=\vec{\gamma}$ does not fix $\vec{\alpha}$ when $\vec{\beta}$ and $\vec{\gamma}$ are known.

The fact that vectors are not generalised numbers makes it impossible to create a new calculus in which vectors shall be the only objects considered. As a matter of fact, the so-called vector-analysis is not a new calculus, but merely a syncopated form of ordinary Cartesian analysis.

Hamilton saw all this, and realised that a true calculus of vectors could exist only within a framework which would have to be provided by a set of generalised numbers; and what he did in discovering quaternions was, precisely, to introduce this framework.

His idea was a very simple one, namely, that the quotient of two vectors was to be regarded as an object of a new kind, to be called a *quaternion*. He then showed that quaternions so defined were generalised numbers, and found that quaternion addition and multiplication satisfied the associative and distributive laws, and the commutative law of addition, but not the commutative law of multiplication—an epoch-making discovery. In terms of quaternions, all the properties of vectors, of rotations in space, etc., can be very simply represented: thus, vectors may be regarded as "quadrantal" quaternions, *i.e.* quaternions whose "angle" is a right angle; and as to rotations in space, we have the simple formula, that if $\vec{a}$ is a vector, then the vector which is obtained from $\vec{a}$ by rotating it about any axis in space, through any angle, is represented by $\vec{qaq^{-1}}$, where q is a quaternion which specifies the rotation.

Both Gibbs and Heaviside spent a good deal of time and energy in attacking quaternions, notably in the controversy of 1892–93; and I think I was justified in describing this energy as "wasted". Their other works show that they were indeed great men, but they did not rise to the idea of treating vectors by means of a calculus of generalised numbers of a new type: all they wanted was some way of writing ordinary Cartesian analysis without putting the axes of coordinates in evidence. Up to a point, this aim is achieved by "vector-analysis": but the scope of "vector-analysis" is, for the

reasons given above, very limited. When I called it a " bastard " derivative of quaternions, I meant that it was the progeny of quaternions, but not in the legitimate line of succession and evolution. I am, etc., E. T. WHITTAKER.

THE RESEARCH METHOD IN TEACHING MATHEMATICS.

To the Editor of the *Mathematical Gazette*.

SIR,—Miss Knowles reminds us of one of the distinguishing advantages of mathematics as a school subject, namely that it is something for a boy to do, rather than something he must learn. This advantage is shared by carpentry, music and drawing, by English if it is taught the right way, and to a more limited extent by foreign languages.

The Spens Committee have missed this vital point, for they seek to reduce the time allowed for the subject, at the same time increasing the field to be covered. They recommend a more descriptive treatment. In other words they would have it become the kind of subject that can be taught by dictating notes, a " crammable " subject, a thing to learn, not a thing to do.

It is significant that " general science " is considered a good choice for a not very able boy who wishes to pass his School Certificate examination. Yours faithfully, E. H. LOCKWOOD.

Felsted School, Ross-on-Wye.

1357. He (the Archbishop's grandfather) goes to Cambridge for study in 1780, but writes in his diary : " The libraries of Cambridge not well supplied with books : no studies in any credit there but mathematical ones." (p. 10.)

1358. She (the Archbishop's mother) taught arithmetic, with very little knowledge of arithmetic herself, by steady repetition. She had a key to the sums in the arithmetic book, giving the answers. If a sum was brought to her and the answer was wrong, she drew her pencil through it and made no further remark. It had to be done again till it was done right. The sum of today was repeated tomorrow, and so on, until perfect accuracy was obtained. —(From a Memorandum by the Archbishop's sister.) (p. 17.)

1359. Euclid was the same. She (his mother) did not understand a word. He began to do so as he advanced in the subject, and could substitute one expression for another, or change the order of letters. She interposed and corrected him. He would reply impatiently " It was all the same." " Say it ", she ordered, " precisely as it is here," touching the book.—(From Memorandum by the Archbishop's sister.) (p. 18.)

1360. . . . The Archbishop told him that on his ninth birthday, to the best of his recollection, after he had gone to bed, his mother happened incidentally to mention to his father that she had carried out his orders to teach the boy Euclid, and that he knew his Euclid. " What! all of it? Can he say any proposition? " " Yes, he knows it all." The father, naturally disbelieving this, had the child woke up, when he repeated, sitting in bed, a long proposition." (p. 18.)

Gleanings 1357–1360 from *Memoirs of Archbishop Frederick Temple*, I. [Per Mr. A. F. Mackenzie.]

MATHEMATICAL NOTES.

1511. *An analytical proof of Pascal's theorem.*

The perusal of Note 1474 has reminded me that it has frequently occurred to me that I have never seen in print a straightforward genuinely analytical proof of Pascal's theorem, such as is given below. Probably nobody prior to Cayley would have expended his energies on a proof of this kind, while later writers would realise that, in general, there was no particular point in doing so. A search through Cayley's works brought to light a paper written a year and a half after he graduated (*Cambridge Math. J.* 4 (1845), 18–20) which was of the nature that I was seeking, but his outlook seemed sufficiently different from mine to justify the present Note.

Given any non-degenerate plane quadratic locus, take as triangle of reference any chord (not a tangent) of the locus and the tangents at the ends of the chord. By choosing suitable multiples of areal coordinates as coordinates, we can write the equation of the locus in the form

$$xz = y^2,$$

the chord being the line $y = 0$. The parametric representation of a point on the locus may then be taken to be $(1, t, t^2)$.

Take now six arbitrary points on the locus with parameters $t_1, t_2, \ldots, t_6$, and call these the points 1, 2, ..., 6. The equation of the chord 12 is

$$xt_1t_2 - y(t_1 + t_2) + z = 0,$$

with a corresponding equation for the chord 45; and hence the point of intersection of the chords 12 and 45 (say the point (12, 45)) has coordinates proportional to

$$t_1 + t_2 - t_4 - t_5, \quad t_1t_2 - t_4t_5, \quad t_1t_2(t_4 + t_5) - t_4t_5(t_1 + t_2).$$

Appropriate changes of suffixes make it clear that the equation of any line through the point (23, 56) may be taken to be

$$A\{xt_2t_3 - y(t_2 + t_3) + z\} = B\{xt_5t_6 - y(t_5 + t_6) + z\},$$

where A and B are not both zero. This line passes through the point (12, 45) when

$$\begin{aligned} &A\{t_2t_3(t_1 + t_2 - t_4 - t_5) - (t_2 + t_3)(t_1t_2 - t_4t_5) + t_1t_2(t_4 + t_5) - t_4t_5(t_1 + t_2)\} \\ &= B\{t_5t_6(t_1 + t_2 - t_4 - t_5) - (t_5 + t_6)(t_1t_2 - t_4t_5) + t_1t_2(t_4 + t_5) - t_4t_5(t_1 + t_2)\}. \end{aligned}$$

This condition reduces to

$$A(t_1 - t_3)(t_2 - t_4)(t_5 - t_2) = B(t_1 - t_5)(t_2 - t_5)(t_4 - t_6);$$

and so, since $t_2 - t_5$ is not zero so long as the points 2 and 5 do not coincide, we see that the line joining (12, 45) to (23, 56) is obtained by taking

$$A = (t_1 - t_5)(t_4 - t_6), \quad B = -(t_1 - t_3)(t_2 - t_4).$$

The equation of the line is therefore

$$\begin{aligned} &(t_1 - t_5)(t_4 - t_6)\{xt_2t_3 - y(t_2 + t_3) + z\} \\ &\qquad + (t_1 - t_3)(t_2 - t_4)\{xt_5t_6 - y(t_5 + t_6) + z\} = 0, \end{aligned}$$

and this immediately reduces to

$$x(t_1t_2t_3t_4 - t_2t_3t_4t_5 + t_3t_4t_5t_6 - t_4t_5t_6t_1 + t_5t_6t_1t_2 - t_6t_1t_2t_3)$$
$$- y\{t_1t_2(t_4+t_5) - t_2t_3(t_5+t_6) + t_3t_4(t_6+t_1) - t_4t_5(t_1+t_2) + t_5t_6(t_2+t_3)$$
$$- t_6t_1(t_3+t_4)\} + z(t_1t_2 - t_2t_3 + t_3t_4 - t_4t_5 + t_5t_6 - t_6t_1) = 0.$$

Since this equation is unaltered by changing the suffixes 1, 2, 3, 4, 5, 6 into 3, 4, 5, 6, 1, 2, it is clear that the line joining the points (12, 45) and (23, 56) must also pass through the point (34, 61); and this is Pascal's theorem.

A moment's consideration shows that the equation of the Pascal line just given is irreducible when the parameters are arbitrary; for the coefficient of z does not factorise, nor is this coefficient a factor of the coefficient of y.

The result of Note 1474, namely that the number of Pascal lines derivable from six points on a conic is not less than 60 when the points are chosen arbitrarily, is clear from the equation of the Pascal line. For, by permuting the order of the suffixes of the parameters, we change the coefficient of z into an essentially distinct function of the parameters and therefore change the equation of the line except when the permutation consists of merely a cyclical interchange, or a reversal of the order, or a cyclical interchange combined with a reversal of the order. These special permutations either leave the coefficients in the equation unaltered or else change the signs of all of them. In other words, given six arbitrary points on a conic, any two different hexagons with those points as vertices have different Pascal lines.

It would be an interesting exercise to enumerate the number of distinct Pascal lines when the conic degenerates into a line-pair, distinguishing between the different distributions of the six points on the two lines, namely, three on each, four on one and two on the other, five on one and one on the other, six on one and none on the other. G. N. W.

1512. *How many Pascal lines has a sixpoint?*

In Note 1474 Professor Neville asks an interesting and fundamental question: can we be certain that the 60 Pascal lines associated with a set of six points on a conic are distinct? It seems worth while to add an algebraic proof to Professor Neville's geometric investigation.

Let the six points have coordinates $(t_i^2, t_i, 1)$, $(i = 1, \ldots 6)$ on the conic $xz - y^2 = 0$. Naming the points according to the suffixes of their respective parameters, the Pascal line of the hexagon 153426 is also the cross axis of the homography on the conic defined by mating the points 123 in order with the points 456. Since this cross axis meets the conic in the double points of the homography; the parameters of the points in which the Pascal line cuts the conic are the roots of the equation

$$at^2 + (b+c)t + d = 0,$$

where
$$at_1t_4+bt_1+ct_4+d=0,$$
$$at_2t_5+bt_2+ct_5+d=0,$$
$$at_3t_6+bt_3+ct_6+d=0.$$

The Pascal line is therefore uniquely determined by the equation

$$\begin{vmatrix} t^2, & t, & t, & 1 \\ t_1t_4, & t_1, & t_4, & 1 \\ t_2t_5, & t_2, & t_5, & 1 \\ t_3t_6, & t_3, & t_6, & 1 \end{vmatrix} = 0. \quad \text{.....(1)}$$

The question now becomes: do we obtain 60 *distinct* quadratics by permuting the suffixes in equation (1) in all possible ways?

Permutation of the rows and columns of (1) leaves the equation unaltered. There are $3!\times 2!$ permutations on the suffixes 1, ... 6 which give the same Pascal line, and therefore there are not more than $6!/12$ distinct lines. There are *exactly* 60 distinct lines if we can show that the only permutations which leave (1) essentially unchanged are the twelve just considered.

Let us suppose therefore that a permutation of the suffixes has changed equation (1) into the equation

$$a't^2+(b'+c')\,t+d'=0 \quad \text{.....(2)}$$

and that this gives the same Pascal line as (1). Then certainly

$$a/d=a'/d' \quad \text{.....(3)}$$

where

$$a=\begin{vmatrix} t_1, & t_4, & 1 \\ t_2, & t_5, & 1 \\ t_3, & t_6, & 1 \end{vmatrix}, \quad d=-\begin{vmatrix} t_1t_4, & t_1, & t_4 \\ t_2t_5, & t_2, & t_5 \\ t_3t_6, & t_3, & t_6 \end{vmatrix},$$

and a', d' are obtained from these determinants by the supposed permutation of the suffixes. Equation (3) is equivalent to

$$ad'-a'd=0. \quad \text{.....(4)}$$

Since $t_1, \ldots t_6$ are assumed to be algebraically independent, the relation (4) between $t_1, \ldots t_6$ must be identically true. Now, a and d are irreducible, and therefore so are a' and d'. It follows that the most general way in which (4) can be satisfied is by assuming that $a'\equiv ka$, $d'\equiv kd$, where k is independent of $t_1, \ldots t_6$, *i.e.* k is a constant.

Noting that the determinant $a=\Sigma t_1(t_5-t_6)$, a simple enumeration shows that the only permutations of the suffixes which convert a into a', where $a'=ka$, are those effected by interchanging rows and columns in the determinant, so that $k=\pm 1$. But these permutations are the twelve considered above. There are therefore 60 distinct lines associated with six general points on a conic.

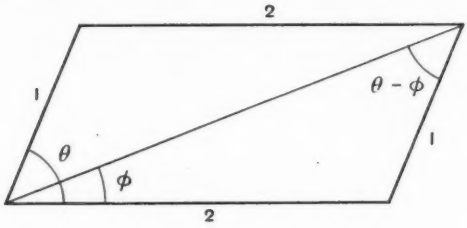
I wish to thank Professor Neville for helping me to clarify my ideas whilst writing this note.

D. PEDOE.

1513. *An American method of approximate trisection of an angle and an extension.*

1. The idea is that since $\frac{1}{3}\alpha = \frac{1}{4}\alpha + \frac{1}{3}(\frac{1}{4}\alpha)$ an approximate method of trisecting $\frac{1}{4}\alpha$ will give a correspondingly *better* relative approximation for the trisection of α itself.

Now the simplest method of roughly trisecting a small angle θ is shown in the figure, where $\phi \doteq \frac{1}{3}\theta$; actually, of course,



$$\cot \phi = 2 \operatorname{cosec} \theta + \cot \theta,$$

and we have the table:

θ	ϕ	$\frac{1}{3}\theta$
0°	0°	0°
10°	3° 20′	3° 20′
20°	6° 38′	6° 40′
30°	9° 54′	10°
40°	13° 5′	13° 20′

Thus there is an error of $\frac{1}{4}$ of a degree when $\theta = 40°$. Since $\theta = \frac{1}{4}\alpha$, the process gives a trisection of 160° with only $\frac{1}{4}$ of a degree error, and of course very much smaller error for any acute angle.

2. *Extension.*

Since $\sum_1^\infty (\frac{1}{4})^n = \frac{1}{3}$, successive division by 4 will alone give successive approximations to trisection.

Thus if $\alpha = 80°$,

$$\tfrac{1}{4}\alpha + \tfrac{1}{4}(\tfrac{1}{4}\alpha) = 25°,$$

$$\tfrac{1}{4}\alpha + \tfrac{1}{4}(\tfrac{1}{4}\alpha) + \tfrac{1}{4}(\tfrac{1}{16}\alpha) = 26°\ 15°, \text{ etc.},$$

and at any stage, the process may be "rounded off" by the 2 : 1 construction of this American method.

Thus if the angle $\alpha = 160°$, take first

$$\tfrac{1}{4}\alpha = 40°,$$

then $\quad \frac{1}{4}(\frac{1}{4}\alpha) = 10°$

then $\quad \phi = 3°\ 20'$ when $\theta = 10°$

$$53°\ 20'$$

With four-figure tables the result is indistinguishable from $\frac{1}{3}\alpha$.

H. V. S.

1514. *Integral-sided geometric figures.*

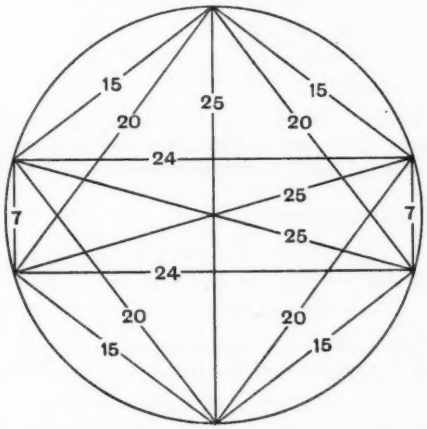
Just as some people love long computations, so others are fascinated by integral-sided geometric figures. Many may have noticed that

$$25^2 = 625 = 400 + 225 = 20^2 + 15^2$$
$$= 576 + 49 = 24^2 + 7^2.$$

This, of course, means that in a semi-circle having diameter 25 units, one can fit two right-angled triangles with integral sides.

Now it is also true that

$$(20 \times 7) + (15 \times 24) = 25 \times 20.$$



Ptolemy's theorem is suggested, giving a cyclic quadrilateral with one of its diagonals and also one of its sides of 20 units. This leads to an isosceles triangle, . . . , and by the time one has finished with this figure it is a cyclic hexagon with all sides and diagonals of integral lengths, and quite small integers, too. Meccano strips can be used with effect.

Does anyone know of others like it? Puzzle makers should enjoy it.

C. DUDLEY LANGFORD.

1515. *The foci of a conic.*

The coordinates of the foci and the lengths of the axes of the central conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (C \neq 0) \ldots\ldots\ldots\ldots\text{(i)}$$

may be found as follows. Referred to parallel axes through the centre, the equation of the conic is

$$ax^2 + 2hxy + by^2 + \Delta/C = 0. \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Taking the foci as (α, β) and $(-\alpha, -\beta)$, and the major axis as $2p$, then (ii) and

$$\{(x-\alpha)^2+(y-\beta)^2\}^{\frac{1}{2}} = \pm 2p \pm \{(x+\alpha)^2+(y+\beta)^2\}^{\frac{1}{2}} \quad \text{.....(iii)}$$

represent the same conic. Upon rationalising, (iii) is

$$x^2(\alpha^2-p^2)+2xy(\alpha\beta)+y^2(\beta^2-p^2)+p^2\{p^2-(\alpha^2+\beta^2)\}=0. \quad \text{...(iv)}$$

Comparing (ii) and (iv),

$$\frac{\alpha^2-p^2}{a}=\frac{\alpha\beta}{h}=\frac{\beta^2-p^2}{b}=\frac{p^2\{p^2-(\alpha^2+\beta^2)\}C}{\Delta}=k, \text{ say.} \quad \text{.....(v)}$$

Then
$$k^2=\frac{(\alpha^2-p^2)(\beta^2-p^2)-(\alpha\beta)^2}{ab-h^2}$$
$$=\frac{p^2\{p^2-(\alpha^2+\beta^2)\}}{ab-h^2}$$
$$=k\Delta/C^2.$$

Hence $k=\Delta/C^2$ or 0.

The possibility $k=0$ can be ignored, since then the numerators of the fractions in (v) must vanish and $\alpha=\beta=p=0$ and these hold only if $\Delta=0$.

To find the foci we proceed thus :

$$(\alpha^2-\beta^2)/(a-b)=\alpha\beta/h=\Delta/C^2. \quad \text{.....(vi)}$$

Thus $C^2\alpha^2/\Delta$ and $-C^2\beta^2/\Delta$ are the roots of

$$\phi^2-(a-b)\phi-h^2=0,$$

and so if ϕ_1 is the root of this equation having the same sign as Δ, and ϕ_2 the root of opposite sign, the real foci of (i) are the two points

$$[\{G \pm \sqrt{(\Delta\phi_1)}\}/C, \quad \{F \pm \sqrt{(-\Delta\phi_2)}\}/C],$$

where the signs before each square root are chosen to be the same if $h\Delta$ is positive and opposite if $h\Delta$ is negative (since $\alpha\beta=\Delta h/C^2$ by (vi)).

To find the lengths of the axes, if q^2 is the square of the minor semi-axis, negative for a hyperbola, $q^2=p^2-(\alpha^2+\beta^2)$.

From (v) $$\frac{\alpha^2+\beta^2-2p^2}{a+b}=\frac{p^2\{p^2-(\alpha^2+\beta^2)\}C}{\Delta}=k,$$

and so $$-(p^2+q^2)/(a+b)=p^2q^2C/\Delta=\Delta/C^2.$$

Hence C^2p^2/Δ, C^2q^2/Δ are the roots of

$$\theta^2+(a+b)\theta+C=0,$$

and the lengths of the semi-axes are

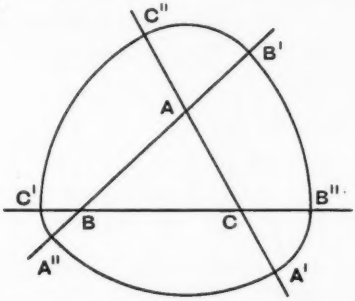
$$\sqrt{(\Delta\theta_1)}/C, \quad \sqrt{(\Delta\theta_2)}/C.$$

R. C. LYNESS.

1516. *On Gleaning* 1325.

With reference to Gleaning 1325, a simple way of describing a curve of constant " diameter " was pointed out to me by Mr. A. G.

Cooper of the Old Students Association, College of Technology, Manchester. It is as follows, and may be known to many readers.



Draw any triangle ABC and produce all its sides. Take any point B' on BA produced and with centre A, radius AB', describe an arc $B'C''$. Then, with centre C, radius CC'', describe an arc $C''C'$; with centre B, radius BC', describe an arc $C'A''$, and so on. The continuous curve thus formed is easily shown to be closed and of constant "diameter", that is, a normal at any point cuts the curve normally at a second point and the segment of the normal intercepted by the curve is of constant length.

Such a curve could not be distinguished from a circle by a pair of calipers alone: calipers with three-point contact would be needed. Cylindrical rollers with such curves as their cross-sections would roll between two parallel planes if the constant "diameters" of the curves were equal.

Mr. Cooper also presented to the mathematics department of the College three steel balls of constant "diameter", constructed in a similar way from a regular tetrahedron: a regular tetrahedron was chosen to secure symmetry about an axis, so that the balls could be turned in a lathe. F. BOWMAN.

1517. *Triangle formulae.*

1. The essentials, but not the details, of the proof of the formulae expressing the tangents of the half-angles of a triangle in terms of the sides of the triangle, given by A. Hinckley (Note 1460, *Gazette*, XXIV, July 1940) appear in Brewster's *Trigonometry*, pp. 55–57.

The following is a modification of Brewster's proof, avoiding the use of the symbol Δ^2, the dimensions of which may present difficulty to the student.

In the triangle ABC (Fig. 1), let Δ denote the area, r and I the radius and centre respectively of the inscribed circle, r_1 and I_1 the radius and centre respectively of the escribed circle opposite A. Then

$$\Delta = rs = r_1(s-a) = \sqrt{\{s(s-a)\, rr_1\}}.$$

Produce I_1D_1 to P so that $D_1P=r$. Then IPD_1D is a rectangle and $\angle IPI_1=90°$; II_1 is the diameter of the circle IBI_1C, therefore P lies on the circumference of this circle, and so we have

$$BD_1=s-c, \quad CD_1=s-b,$$

and hence

$$rr_1=(s-b)(s-c).$$

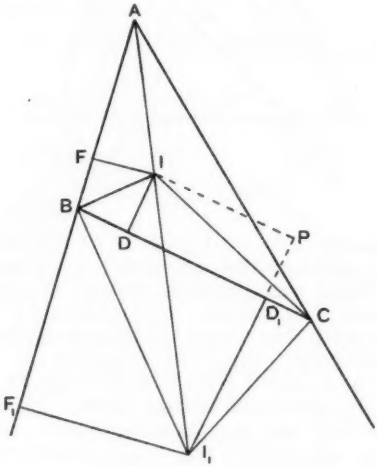


FIG. 1.

Thus

$$\Delta=\sqrt{\{s(s-a)(s-b)(s-c)},$$

$$r=\Delta/s,$$

$$\begin{aligned}\tan \tfrac{1}{2}A &= r/(s-a)\\ &= \Delta/s(s-a)\\ &= \sqrt{\{(s-b)(s-c)/s(s-a)\}}.\end{aligned}$$

2. It does not appear to be well known that Napier's tangent rule can be proved by using the above figure as follows:

$$\begin{aligned}\angle BI_1F_1 &= \angle BI_1D_1\\ &= 90°-\tfrac{1}{2}(180°-B)\\ &= \tfrac{1}{2}B.\end{aligned}$$

Since IBI_1C is cyclic,

$$\begin{aligned}\angle BI_1I &= \angle BCI\\ &= \tfrac{1}{2}C.\end{aligned}$$

Thus

$$\angle F_1I_1A=\tfrac{1}{2}(B+C),$$

$$\angle II_1P=\tfrac{1}{2}(B-C),$$

$$DD_1 = (s-c)-(s-b)$$
$$= b-c$$
$$= IP.$$

Hence
$$\tan \tfrac{1}{2}(B-C) = (b-c)/(r_1+r),$$
$$\tan \tfrac{1}{2}(B+C) = s/r_1 = (s-a)/r$$
$$= (2s-a)/(r_1+r)$$
$$= (b+c)/(r_1+r).$$

Thus
$$\frac{\tan \tfrac{1}{2}(B-C)}{\tan \tfrac{1}{2}(B+C)} = \frac{b-c}{b+c}$$

3. The following proof of Napier's tangent rule is given by F. L. Griffin in his *Introduction to Mathematical Analysis*, but that author does not seem to realise the generality of the proof as he proposes a special proof for an obtuse-angled triangle.

In the triangle ABC (Fig. 2) let $A > B$. Construct the angle $ABD = A$. Produce AC (or CA) by a length x to meet BD (or DB)

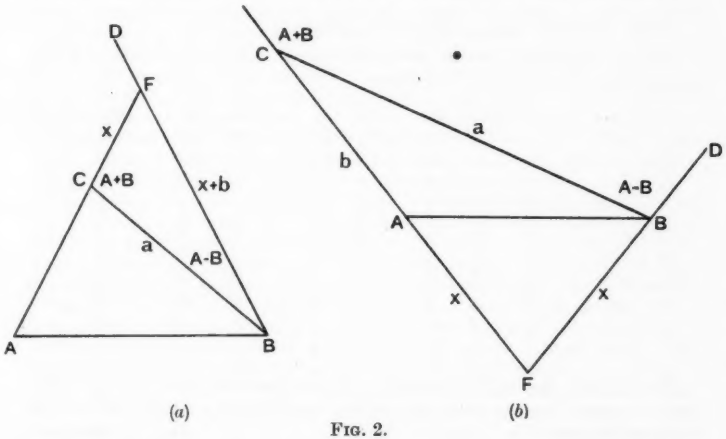


FIG. 2.

in F. Let s' be the semi-perimeter of the triangle BCF and r' the radius of the circle touching the arms of the angles $A+B$ and $A-B$, *i.e.* the inscribed circle in Fig. 2*a* and an escribed circle in Fig. 2*b*.

$$\tan \tfrac{1}{2}(A-B) = r'/(s'-x)$$
$$= r'/\{x + \tfrac{1}{2}(a+b) - x\}$$
$$= 2r'/(a+b);$$

$$\tan \tfrac{1}{2}(A+B) = r'/\{s' - (x+b)\}$$
$$= r'/\{x + \tfrac{1}{2}(a+b) - (x+b)\}$$
$$= 2r'/(a-b).$$

The required formula is obtained by division. RAYMOND SMART.

1518. *A theorem in solid geometry.*

The following proof of a standard theorem has been given to me by Mr. I. W. Stoddart. Is it new?

ABCD is a tetrahedron in which DA is perpendicular to BC, and DB is perpendicular to AC. To prove that DC is perpendicular to AB.

Through A, B, C draw lines parallel to BC, CA, AB to form the triangle XYZ. Then A, B, C are the mid-points of YZ, ZX, XY. Since DA is perpendicular to BC, DA is perpendicular to YZ.

Therefore	$DY = DZ$.
Similarly	$DX = DZ$.
Hence	$DX = DY$.

Therefore DC is perpendicular to XY and therefore perpendicular to AB. C. V. DURELL.

1519. *An extension of Pascal's triangle.*

Pascal's triangle can be extended to give the coefficients of the expansion of $(1+x)^{-n}$, as follows:

$(1+x)^{-5}$	-	1	−5	15	−35	70	−126	210	−330	...
$(1+x)^{-4}$	-	1	−4	10	−20	35	−56	84	−120	...
$(1+x)^{-3}$	-	1	−3	6	−10	15	−21	28	−36	...
$(1+x)^{-2}$	-	1	−2	3	−4	5	−6	7	−8	...
$(1+x)^{-1}$	-	1	−1	1	−1	1	−1	1	−1	—
$(1+x)^{0}$	-	1	0	0	0	0	0	0	0	...
$(1+x)^{1}$	-	1	1	0	0	0	0	0	0	...
$(1+x)^{2}$	-	1	2	1	0	0	0	0	0	...
$(1+x)^{3}$	-	1	3	3	1	0	0	0	0	...
$(1+x)^{4}$	-	1	4	6	4	1	0	0	0	...
. . . . .										

To extend the table upwards, all we have to note is that any entry is equal to the entry below minus the entry to the left of the required entry. G. A. GARREAU.

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REVIEWS.

A Mathematician's Apology. By G. H. HARDY. Pp. viii, 94. 3s. 6d. 1940. (C.U.P.)

In 1907 there were no Archimedeans at Cambridge, nor was there a mathematical club in Trinity, but one Sunday evening in a philosophical essay society which owed its existence to the hospitality of the Professor of Arabic a junior lecturer defended his vocation. So when a third of a century later Professor Hardy begins his apology by bemoaning the melancholy experience of finding himself writing "about" mathematics, we are not all wholly deceived. He has always enjoyed arguing "about" mathematics, and surely such talk is as legitimate a relaxation for a professional mathematician even in his most fertile years as chess or cricket.

The greater part of this booklet deals with the nature of mathematics and with the relation of mathematics to the life of the community. The "real" mathematics of the great mathematicians is contrasted with the "trivial" mathematics which is developed for the sake of applications. Had he not retired behind the impervious cover of the admission that "the most practical of mathematicians may be seriously handicapped if his knowledge is the bare minimum which is essential to him" (p. 74), Professor Hardy might be charged with underestimating the range of "real" mathematics which has been found "useful": conformal transformations and the properties of geodesics are far above the datum line of his academic contempt. But when every allowance is made, the demands of "trivial" mathematics are wholly insufficient to justify the study of "real" mathematics. What is more, the creative mathematician disclaims passionately, resents bitterly, the suggestion that a justification of his lifework can be found in the ugliest and most despicable outgrowths from the vulgar roots of his subject. If pictures and statues have no intrinsic value, the painter will not accept patronage as an interior decorator or the sculptor as an architect's assistant, and if we loathe music, we must not hope to placate the musician by our gratitude when we find that it is profitable to milk cows to a seductive melody.

The apology for a life devoted to "real" mathematics hangs therefore on the aesthetic value of the mathematics itself. Every mathematician believes this in his heart, and Professor Hardy, in the perfect prose of which he is master, defends and elaborates our common conviction. He even succeeds in describing in simple language some of the constituents of mathematical beauty.

But Professor Hardy's creed is vitiated by an irreconcilable contradiction, for it includes one article which would be disputed by many artists and which is, I hope, peculiar to himself among mathematicians. "Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune" (p. 26); "The best [mathematics] may, like the best literature, continue to cause intense emotional satisfaction to thousands of people after thousands of years" (p. 71). On the other hand, "Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or the desire to create" (p. 83). That is to say, when Littlewood or Ramanujan showed him a theorem, the only reaction possible to Professor Hardy was "What can I make out of this myself? How does it help me in my own researches?" The spontaneous enjoyment of a beautiful result for its own sake, the generous congratulation of a friendly investigator, the ready appreciation of achievement in a branch of mathematics which he has not made his own—these, if we are to believe him, are outside Professor Hardy's experience. I for one do not believe him "(nor should I think the better of him if I did)" (p. 19). E. H. N.

An Introduction to the Theory of Newtonian Attraction. By A. S. RAMSEY. Pp. ix, 184. 10s. 6d. 1940. (Cambridge)

Teachers of Applied Mathematics will welcome the appearance of this book, in which the standard theorems of the subject are collected together and are stated concisely in a manner which makes the book easy to read. The examples used in the text to illustrate these theorems will be found very useful by a student reading the book without the help of a teacher. The chapters of the book deal in succession with (1) the necessary theorems in Pure Mathematics, (2) the attraction and potential at external and then (3) at internal points of a continuous distribution of matter, (4) the theorems of Laplace, Poisson, and Gauss, (5) Green's theorem, (6) harmonic functions and finally with (7) the attraction and potential of ellipsoids. The ground covered is that of an honours course, but the second and third chapters are intended for students reading for an ordinary degree; the examples are numerous and the division of each set into two parts, easy and harder examples, is, so far as can be judged without more detailed examination, satisfactory. It is interesting to note the use that has been made of London University examination papers which provide a considerable number of the easier examples in Chapters II and III besides being well represented in the other chapters.

The subject of elementary gravitational theory, having reached a high standard of development by the beginning of the nineteenth century, is now in a stable state, and it is therefore unlikely that a reviewer can find much to criticise in the treatment given in this book though teachers will probably wish to alter the order of the exposition in some cases and to make small changes in other cases, especially in the first four chapters. The following remarks concern matters of relatively small importance, but may be worth mentioning as showing how satisfied this reviewer is with the book as a whole.

In Chapter I, dealing with preliminary mathematics, there are two proofs of the theorem connecting surface and volume integrals; it is, perhaps, not sufficiently emphasized that the proof given in 1·7 is a deduction from the definition

$$\operatorname{div} \mathbf{A} = \lim_{\delta v \to 0} \frac{\int A_n \, dS}{\delta v},$$

used in the treatment given in many modern books on vector analysis, and that the first proof in 1·3 is used to prove that this limit exists and is unique. In Chapter III there is no need to use a cavity in order to obtain the attraction and potential of a uniform solid sphere at internal points, since it has been proved in 1·8 that the integrals to be dealt with are convergent. In any case it requires further consideration to show that the proof given in 1·8 can be applied to the particular cavity chosen, which is spherically symmetrical, since a sequence of such cavities of decreasing volume does not close in on the point from *all* sides. If it is intended that Chapter III should be read independently of Chapter I it would be preferable either to give some explanation of the necessity of using cavities in defining the potential and attraction at internal points or to give some indications to show that the integrals concerned are convergent; these approaches to the work of Chapter III are essentially the same since cavities are the physical counterpart of the ideas used in the definition of the value of an infinite integral. Another method, which follows the line of least resistance and is probably most suitable for the "pass" student, is to use the results to show their consistency with the theorem of the continuity of the potential (suggested by physical considerations) and with the fundamental theorem that the attraction is the gradient of the potential.

The use of geometrical proofs, following Newton, though interesting historically, is of limited application and the very useful method, due to Maupertuis, of using as the variable of integration the distance of the point P at which the potential or attraction is required from the rim of an elementary zone perpendicular to OP, where O is the centre of the spherical shell, deserves more notice than the few lines given to it at the end of the second example in 3·9. The "pass" student in particular feels more at home with proofs using the Calculus, with which he is familiar in dealing with the determination of centres of mass and moments of inertia, than with geometrical methods which he would find difficult to apply to other cases, *e.g.* that of finding the attraction and potential of a portion of a uniform spherical shell.

There is no discussion of the mechanical force on a surface layer and consequently no examples on the calculation of such forces. Probably this omission is due to the fact that this point is dealt with in the author's *Electricity and Magnetism* where it is naturally of more importance than in gravitational theory.

In using Legendre's theorem by which the potential V of matter having axial symmetry can be determined when it is known at points on the axis of symmetry, there is sometimes an apparent discontinuity in the potential and its derivative, as in the example of 6·32 dealing with a circular disc. In this case V and $\partial V/\partial r$ are apparently discontinuous when $r=a$, the radius of the disc, their values being given by different expressions for $r < a$ and $r > a$. This point was investigated by Bromwich (*Phil. Mag.* 2, p. 237, 1910), whose work contains a detailed discussion of this example which shows that the discontinuity does not really exist.

With regard to the last chapter on the attraction and potential of ellipsoids, the argument set out in the sequence of proofs is somewhat shortened and simplified by using throughout the potential instead of the components of attraction, *e.g.* in proving Ivory's theorem and in deducing MacLaurin's theorem from it. This change was introduced by Chasles and his treatment is reproduced in Thomson and Tait, *Natural Philosophy*, Vol. II.

There is one question of general interest on which teachers will have definite views, viz. how far, if at all, is it desirable to include something approaching a rigorous treatment of the fundamental theorems of gravitational theory? It may be argued that students have enough practice in the ϵ, δ technique in Pure Mathematics, and that in Applied Mathematics their time would be better spent in becoming acquainted with the experimental side of a subject and in understanding the physical arguments often used to "prove" a mathematical theorem. The author appears to be in agreement with this view as, apart from the discussion of the convergence of a certain type of volume integral in the first chapter, with a reference to Leathem's Cambridge tract, no attempt is made to deal with any theorems on similar lines or to explain why the proofs of some theorems, *e.g.* Poisson's equation $\nabla^2 V = -4\pi\gamma\rho$, should be regarded with suspicion. On the other hand, in gravitational theory there is not the same frequent appeal to experiment and physical argument as in electricity and magnetism, and thus the former theory is essentially a logical development from a single axiom, the law of gravitation, so that a rigorous treatment of its fundamental theorems would have made the book complete in itself. Even if rigorous proofs of these theorems are not asked for in examinations, this is no reason why they should not find a place in a book written for honours students, and it was a disappointment to find that the author had not taken the opportunity of making his book the first British textbook to include a satisfactory treatment of the foundations of the theory of the potential, so far as that can be done without taking up a disproportionate amount of space.

As regards volume distributions, the theorems referred to above are that the attraction is the gradient of the potential and that the second derivatives $\partial^2 V/\partial x^2$, etc., exist, from which result Poisson's equation can be established. The proof of the first theorem, as given by Poincaré, *Théorie du Potentiel Newtonien*, should not be beyond the capacity of a good honours student who has to deal with more difficult ideas in analysis, and a proof on similar lines can be given for the second theorem, both proofs being subject to certain restrictions on the function which represents the volume-density. Alternatively a method, due to Gauss and used throughout his tract by Leathem, which is based on the theorem connecting surface and volume integrals, can be used to prove these two theorems. There are similarly two important theorems dealing with surface distributions which require consideration and demand rigorous proof; they are the continuity of the limits of the tangential components and the discontinuity of the limits of the normal components, as a point on the surface is approached from both sides, and the connection of these limits with the corresponding components *at* a point on the surface. If one is content to make simplifying but at the same time reasonable assumptions as to the type of surface and the surface-density, the proofs are not so difficult as many of the well-known theorems in analysis. It is true that the consideration of these four theorems, together with introductory matter and illustrative examples, would seem, if dealt with on the lines given by Poincaré or Leathem, to require more space that could be given to it without overloading this part of the book. The proofs can, however, be simplified and can be expressed concisely by assuming, as is naturally a reasonable assumption, a knowledge of the necessary theorems of analysis. A short additional chapter dealing with these theorems would go a long way towards satisfying any doubts that an enquiring student might have as to the foundations of the subject and would thus add greatly to the usefulness of the book.

With the publication of this book Mr. Ramsey has now completed a series of textbooks written in the last twelve years dealing with Statics, Dynamics, Hydrostatics, Advanced Dynamics, Electricity and Magnetism, and Gravitational Theory. As at an earlier date he had written a book on Hydrodynamics, Mr. Ramsey has thus covered the range of Applied Mathematics as required for an honours course in most universities. Mr. Ramsey and the Cambridge University Press will have the information on which a quantitative estimate of the success of this series of books can be based—we know that there can be very few universities in which the books are not used, and the first three books on the list must be in continual use by scholarship candidates in schools.

In the teaching of Applied Mathematics there cannot be the large variations due to different points of view which are found in books on other branches of knowledge; the principles, having been established many years ago, can be found presented in numerous standard books, such as those by Routh, Thomson and Tait, Maxwell, J. J. Thomson, Lamb, to mention only some of those with which we are all familiar, and the textbooks based on these treatises differ chiefly in the choice of the theorems to be regarded as fundamental, in the order of these theorems and in small variations in their proofs due to the particular author. What does change with time is the type of natural phenomena to which these principles are applied; mechanical developments resulting in the construction of aeroplanes and motor cars and physical experiments leading to the formulation of new theories have both influenced considerably the choice of examples used to illustrate the principles of the subject. It is this fact that makes it essential that our textbooks should be kept up to date, and we are very grateful to Mr. Ramsey, not only for providing

teachers with almost completely new sets of problems based on recent examination papers, but also for incorporating into the " book-work " the new methods that have been evolved for dealing with the new type of problems. As teachers we all know the stimulus provided by being faced with a problem which opens up before us new ideas and requires us to consult books which we had not known previously or with which we had been only slightly acquainted.

The task of a teacher is so much lightened by the use of Mr. Ramsey's books that the least we can do to show our appreciation of his work is to congratulate him on its completion and to express the hope that in the near future he will find it necessary to provide new editions for a new generation of students.

H. R. Hassé.

Mathematical Methods in Engineering. By Th. von Kármán and M. A. Biot. Pp. xii, 505. 33s. 1940. (McGraw-Hill)

Anyone familiar with the contributions which the authors have made to knowledge in various fields of applied mechanics would expect to find this book vital, stimulating, and full of good things, and he will not be disappointed. Most books on mathematics for technicians introduce applications as and when they are appropriate in a systematic course of mathematics ; our authors claim to have adopted the converse plan, building up the book round groups of engineering topics, introducing and explaining the mathematical technique as it is required. How this is done may best be indicated by the analysis of a typical chapter.

Chapter VII is entitled " The Differential Equations of the Theory of Structures ". After a short introduction, § 1 establishes the differential equation for the shape of a string under tension and transverse load, and builds up the general integral for a distributed load ; the case of a concentrated load is dealt with as a special case of a large load over a small segment, and the result is then shown to be a statement of the equilibrium condition at the load point. This leads to the influence (Green's) function, and so back to an alternative formulation of the displacement formula for a continuous load. § 2 deals with a string subject also to continuous elastic support. § 3 gives the general equations for deflection, slope, bending moment and shearing force in elastic beams. In § 4 the equation for the beam " on an elastic foundation " is introduced and solved (by the help of the fourth roots of -1) and the solution for a concentrated load on an infinite uniform beam is discussed in detail. The effect of a stiffener ring on a pipe subject to internal pressure is then dealt with ; this needs a particular integral as well as the complementary function, for the same differential operator. § 5 treats the simple deflection theory of the suspension bridge (an elastic beam supported by a flexible cable), and gives several methods of approximating to the additional tension due to a live load. § 6 is introductory to the study of vibrations of continuous systems, and shows how the partial differential equations of such problems can be reduced to ordinary differential equations for the " amplitude function ". § 7—Harmonic Vibration of a String under Tension—introduces the student to characteristic numbers and functions, associated with normal modes of vibration. The behaviour of the equation of this section is contrasted with that of the equation of § 2, and the difference is shown to be due to the infinite set of zeros of the general solution, as against the single zero. § 8 deals with the transverse vibration of beams and the whirling of shafts, and various methods of approximating to the roots of the transcendental equations which arise are exhibited. In § 9 the shaft with a pulley at the centre is considered in detail, and mathematical methods of approximating to the pulsatance when the mass of the

pulley is either large or small compared with that of the shaft are shown to be susceptible of a physical explanation. § 10 discusses the forced vibrations of a cantilever; infinities associated with resonance appear. § 11 gives the Euler theory of struts, but in addition the connection between the critical load so determined and stability is shown. In § 12 the critical load of a tapered strut (requiring the solution of a differential equation with variable coefficients), and the greatest length consistent with stability of a vertical column under its own weight, are found; more approximations and limits, and the root-squaring process applied to a truncated series solution of the second problem. In § 14 the buckling of an elastically supported beam is discussed. § 15 deals with combined transverse and axial loads, the example chosen being an aeroplane spar. The differential equations for deflection and bending moment are integrated, and the solution of the latter interpreted geometrically to give the Howard circle diagram. Both analysis and geometry are shown to yield a critical load, the same as in the absence of transverse load. § 16 discusses in general the " boundary problems " of this chapter, contrasting them with those of the theory of vibrations (" jury " as against " marching " problems, in the terminology of L. F. Richardson), and discussing further the differences between solutions, and possibilities of non-zero solution, of homogeneous and non-homogeneous linear differential equations. § 17 illustrates in the case of the transverse vibrations of a stretched circular membrane the iterative method of successive approximation to characteristic numbers. In this closely woven fabric the warp of engineering and the weft of mathematical theory make a pattern to which the above summary can do but scant justice.

As for the other chapters, most of which have a structure similar to that of VII, space forbids more than an indication of their content. The first three chapters are mathematical, and introductory. Of Chap. I (on Linear Differential Equations) and Chap. III (General Principles of Dynamics) it would be easy to say that they are of no use to the student unfamiliar with their content, and useless to the expert. They are convenient for reference back in the later chapters, and are supplemented (as is every chapter) by well-chosen references to other books. Chap. II—Some Information concerning Bessel Functions—is an admirable short introduction to the elementary properties of these functions, likely to be really serviceable to many students. Chap. IV deals with various problems of practical dynamics, and introduces elliptic integrals, and the equations of motion for projectiles under air resistance, and of an aeroplane. Chap. V deals with undamped oscillations; various methods of solving the algebraic " frequency equations " are given and illustrated. Chap. VI treats damped oscillations; discussion of stability, in the case of a top and of an aeroplane, leads to the necessity of determining complex roots of algebraic equations. In Chap. VIII we are introduced to Fourier series and integrals applied to structural (*i.e.* " boundary ") problems; at the end of the chapter the Rayleigh energy method, and the Rayleigh-Ritz method, are introduced. Chap. IX is devoted to " steady state " oscillations, using the complex exponential and the complex form of the Fourier expansion. Chap. X deals with transient phenomena in mechanical and electrical systems. The (so-called) operational calculus is approached via the Fourier integral, and we meet Carson's integral equation, the Laplace transform, the expansion theorem, Duhamel's integral and Borel's theorem, applied to a series of examples. The Mellin theorem is just round the corner, but functions of a complex variable are outside the scope which the authors have set themselves. This chapter is extremely good, in its comprehensiveness and the closeness of its argument, linking together all the ways of regarding or approaching the " operational calculus ". The final chapter is devoted to

finite difference equations, applied to mechanical problems (*e.g.* continuous beams) and to electrical networks. Every chapter is followed by a selection of problems, and by references to aid the further reading of students. At the end we come to a section on " Words and Phrases "—being definitions and explanation of technical mathematical terms—answers to the problems, and a thirty-column index which should make reference easy.

Specially to be commended are, the care which has evidently been taken over the arrangement of the topics in the various sections and in the choice of illustrative examples ; the variety and vitality of these examples and of the sets of problems (only severe self-denial prevented the writer from delaying this review while he worked some of them!) ; the manner in which after results (general or particular) have been obtained by a mathematical argument, the physical meaning of the result, and often of the steps in the argument, are so clearly shown ; and the quotations which head the chapters.

With so many good things it may seem ungracious to complain of occasional over-sophistication (the *first* example of the calculation of Fourier coefficients devotes more space to " tricks " enabling the range of integration to be reduced to a quarter-period than would be used in a direct application of the general rule), or to point out that the physics of the " beam on an elastic *foundation* " is faulty, when such good didactic use is made of the differential equation. One serious criticism must be levelled, however. In electrical theory, impedance is universally defined as the ratio of E.M.F. to *current* ; in the scheme of electro-mechanical analogy generally adopted—and emphasised in this book—mechanical impedance would be the ratio of force to *velocity*. In (practically) undamped mechanical systems the impedance so defined would be a (practically) pure reactance ; this may be inconvenient, and for more reason than that a ratio of force to *displacement* may be preferable. But to call such a ratio a mechanical *impedance* is to cause needless confusion—especially as acoustical impedance *is* defined in terms of velocity and not displacement. The crime is the worse, inasmuch as Biot has introduced the term " dynamic modulus " to denote the ratio of the peak value of a periodic force to the amplitude of the forced vibration it produces (a very useful concept and term) so that there is no need for the term impedance to be used with an unusual significance.

Type, paper and binding are of the usual McGraw-Hill standard, but a little more intelligent co-operation between authors and compositors would have improved the appearance of many pages, reduced the number of " leads " in setting up the type, and reduced appreciably the weight of the volume.

W. G. B.

Science since 1500. By H. T. PLEDGE. Pp. 357. 7s. 6d. 1939. (H.M. Stationery Office)

In times past the arts and sciences were patronised by princes. The modern millionaire has also done his part, but private benefaction cannot altogether replace royal or state patronage, so it is pleasing to find that the democratic state is aware of its duty in this respect. The present volume is worth much more than the price charged for it by H.M. Stationery Office, and is an admirable product of a system by which the museums are linked on the one hand to the Board of Education and on the other to London University.

The subject is a big one and it is a high tribute to the author to say that he has presented a satisfactory conspectus, without getting lost in the details either of biography or technical development. His point of view is neither that of Professor Hogben, for whom social needs are the primary concern, nor that of Sir William Dampier, who writes the history of science against a background of more intellectual ideas. He takes a middle course, writing

mainly of the technical tradition, but setting his history against the human and economic factors of the times.

This aspect of the different centuries is described in a number of "background studies", which, together with the "concluding discussion", form a series well worth reading for its own sake. In the course of it we may observe that the advancement of science has followed economic prosperity to a greater extent than is sometimes realised, a fact which is well brought out by a series of maps showing the birthplaces of scientists. There are also charts to show the handing on of the tradition from teacher to pupil.

All this is the geography of the subject, and it is interesting enough to follow science from Italy to the Baltic, to Holland and England, and finally to Germany. But other influences must be counted. There is Paris, the intellectual centre, and there are the courts and governments. Industry has been a less direct but important stimulus to research and has in turn received benefit. Instruments, the product of a union of industry and science, have led the way to further advance. The clock leads inevitably to the all-pervading idea of periodic motion; glass leads successively to the telescope, the microscope and the chemistry of gases; pumps, likewise, to steam-engines, thermodynamics and vacuum tubes. Symbols, too, must play their part, and behind all are the ideas, some of which run right through from ancient Greece to modern times. Such, for instance, is the idea of the physical, a less primitive one than might be supposed, for without it, heat was thought to be an element, optics and astronomy were mere exercises in geometry, and substances having the colour of gold were thought to be steps towards transmutation.

Closely akin is the idea of "influences" and "effluvia", developing along one road into bacteriology, along another into aether and field-theories. More abstract and more fundamental are the twin opposites, continuity and discreteness, and the interesting suggestion is made that the idea of continuity is likely to be uppermost in periods preceding a great synthesis, while that of discreteness more often possesses the mind when the results and limitations of such a synthesis are being examined.

Finally there is the idea of space, which may be viewed either as "elbow-room" or as a method of ordering. In the days of the great explorations it was natural to think chiefly of elbow-room, but now that we know that space both on this planet and elsewhere is limited, our thoughts, whether in geography or in pure mathematics, are turned more towards "order".

These ideas run through the book and bind together the chapters on the separate subjects. The central object of the book is to show the inter-relation of the sciences rather than to give a detailed history of each, but it is nevertheless divided for convenience into chapters dealing with particular subjects. At any given time there are usually two or three sciences which are at peaks of development, while others may be producing ideas which will not germinate for centuries. Advantage is taken of this in the plan of the book, which is not so much to consider the whole of science period by period, but to treat in each chapter of one major advance, its precursors and its effects. By this means duplication and kaleidoscopic disorder are avoided, a sense of perspective is retained, and the whole book is made clear and readable. It is well worth the reading, too. E. H. L.

The Special Theory of Relativity. By H. DINGLE. Pp. vii, 94. 3s. 6d. 1940. Monographs on physical subjects. (Methuen)

This book begins in conventional fashion with a carefully illustrated statement of the Principle of Relativity. Then follows a chapter which includes descriptions of the Fizeau and Michelson-Morley experiments. It is at the next stage, that is to say, the explanation of the result of the latter, that the

author claims to adopt a new treatment. This explanation consists of asserting that the quantity which is of fundamental importance in physical laws is not the length l of an object, but $l\sqrt{\{1-(v^2/c^2)\}}$, where v is the velocity of the object relative to the particular frame of reference in use. From this it is deduced that time must be considered similarly, and the equations of the Lorentz transformation follow without further difficulty. From this stage onwards the treatment once more becomes conventional. The generalisation of the laws of mechanics is treated in some detail, and Maxwell's equations receive brief mention. The last chapter is devoted to certain aspects of the General Theory.

The exposition as a whole is very clear, and certain points which might be expected to cause difficulty to the reader are carefully explained, often with the help of well chosen analogies. B. M. B.

The Tutorial Algebra. By W. BRIGGS and G. H. BRYAN. Fifth Edition, revised and rewritten by G. WALKER. Vol. I. Pp. vii, 457. 7s. 6d. 1940. (University Tutorial Press)

"*The Tutorial Algebra*, which for many years has been regarded as a standard work and much appreciated by students and teachers alike, was originally written as a textbook suitable for the Intermediate and Degree examinations of the University of London and other examinations of similar standard. In view of changes which have taken place in recent years in these various examinations and in teaching methods, it was decided that the work should be subjected to a complete revision. It has in consequence been replanned, rewritten and extended in accordance with modern conceptions and teaching methods, and present-day examination requirements. Because of the large amount of new matter incorporated, the present edition is issued in two volumes. Volume I covers the algebra required by the syllabuses of the Intermediate examinations of the University of London, with the exception of Intermediate Economics. Taken in conjunction with Volume II, the syllabus of the latter examination, as well as that of Higher School Certificate examinations, are treated in full."

The foregoing extract from the Preface indicates clearly the aim of this volume, and it seems almost presumptuous to make comments on a work which is in such a full flood of success. But the present reviewer (to his shame be it said) has never seen the original book, and is therefore able to consider this edition entirely without prejudice.

The new work opens with a chapter on Theory of Indices in which reliance is placed on the Principle of the Permanence of Equivalent Forms (with the last four words in heavy type). It seems clear that, among other purposes, this book is written for those who may have to prepare for their examinations by private study alone; and so it is a great pity to be so heavy handed with them. Besides it is not necessary to use a mumbo jumbo phrase, and it is wrong to say that "the removal of the limitations is not *incompatible* (author's italics) with the truth of the law". The fact is that the new doctrine must include the old as a particular case. The right approach is to define a^n when n is a positive integer, deduce the law of indices by counting the a's; and then to say that so far a^0, $a^{\frac{1}{2}}$ a^{-1} have no meaning at all. We are going to invent a meaning and to do so we use what we have just found out about our positive integral indices. Someone once said that God made the integers; and if this is so, it is equally true that man made the indices—fractional and negative.

Next comes a chapter on Surds and Imaginary Quantities, and it is satisfactory to note the firm statement for this stage: "it is assumed that such imaginary numbers obey the same fundamental laws of algebra as do real

numbers ". A chapter of miscellaneous propositions, including the consideration of simple partial fractions and the remainder theorem, is sandwiched between chapters on ratio, proportion and variation, and chapters on equations in one, two and three variables.

We then come to the very important chapters on the idea of a limit and fundamental ideas on convergence and divergence. In these the limit ideas of zero and infinity are very carefully explained, and there are excellent diagrams to illustrate convergent, oscillating and less half-hearted divergent series. These two chapters amply support the claim in the Preface regarding modern conceptions and teaching methods.

The volume continues with chapters on arithmetic and geometric progressions, and other series; theory of quadratic equations and expressions, and of rational functions, with many excellent diagrams and hints in the worked-out examples. After chapters on logarithms, permutations and combinations there is one on mathematical induction with which very special pains have been taken; for as the author says it is " a branch of the subject which offers difficulty to many students ". The last two chapters are on the Binomial Theorem for positive integral index, and on Interest and Annuities. In the former it is gratifying to note the introduction of the compact $\binom{n}{r}$ notation.

This is a long book, and, with the exception noted at the beginning of this review, it is very carefully and clearly written. There are 1177 examples to be worked with answers to them at the end of the book. N. M. G.

Number. The Language of Science. By T. DANTZIG. Second edition. Pp. x, 320. 10s. 1940. (Allen and Unwin)

The main text of this useful popular account of the development of the number-concept is unchanged. About sixty extra pages are included, however, containing twenty-six short appendices and a list of suggestions for further reading. The appendices have their origins in requests from various readers of the first edition for amplification or clarification of points connected with the history, the technique or the philosophy of number-theory. There is, for example, a neat proof of the impossibility of trisecting an angle by ruler and compass construction, and an interesting table of the changes in decimal notation. Some of the appendices are reprinted from other works by Dr. Dantzig.

Appendix 2, " On matching and grouping ", refers to a " conundrum . . . in vogue a few years ago ": " If there are more people in this world than hairs on any human head, then there are at least two people in the world who have an equal number of hairs." This, of course, rests on Dirichlet's principle that if $n+1$ objects are to be placed in n boxes, one box must contain at least two objects. But the principle and the conundrum itself are much older than this, since we find * Nicole, one of the authors of the Port Royal *Logic*, putting the statement and its proof to Madame de Longueville, in the seventeenth century.

Remarks on binary, quinary and vigesimal systems of numeration, in Chapter I, prompt the query whether or no it is still possible, as it was some few years ago, to hear, in some of the remote North Country dales, a shepherd counting his sheep somewhat after this fashion: " Yain, tain, . . . disc, yain-disc, . . . bumpitt, bumpitt-yain . . . ", that is, " One, two, . . . ten, eleven, . . . fifteen, sixteen . . . ". There would seem here to be some curious mixture of denary and quindenary scales. T. A. A. B.

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